Multi-Armed Bandits

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CS/Stat 184(0): Introduction to Reinforcement Learning

Fall 2024

Today

- Feedback from last lecture
- Recap
- Multi-armed bandit problem statement
- Baseline approaches: pure exploration and pure greedy
- Explore-then-commit

Feedback from feedback forms

1. Thank you to everyone who filled out the forms!
2. Examples of complex algorithms

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Iterative LQR (iLQR)

Recall $x_0 \sim \mu_0$; denote $\mathbb{E}_{x_0 \sim \mu_0}[x_0] = \bar{x}_0$

Initialize $\bar{u}_0^0, \ldots, \bar{u}_{H-1}^0$, (how might we do this?)

Generate nominal trajectory: $\bar{x}_0^0 = \bar{x}_0, \bar{u}_0^0, \ldots, \bar{u}_h^0, \bar{x}_{h+1}^0 = f(\bar{x}_h^0, \bar{u}_h^0), \ldots, \bar{x}_{H-1}^0, \bar{u}_{H-1}^0$

For $i = 0,1,\ldots$

For each h, linearize $f(x, u)$ at $(\bar{x}_h^i, \bar{u}_h^i)$:

Note that although true f is stationary, its approximation f_h is not

$$f_h(x, u) \approx f(\bar{x}_h^i, \bar{u}_h^i) + \nabla_x f(\bar{x}_h^i, \bar{u}_h^i)(x - \bar{x}_h^i) + \nabla_u f(\bar{x}_h^i, \bar{u}_h^i)(u - \bar{u}_h^i)$$

For each h, quadratize $c_h(x, u)$ at $(\bar{x}_h^i, \bar{u}_h^i)$:

$$c_h(x, u) \approx \frac{1}{2} \begin{bmatrix} x - \bar{x}_h^i \\ u - \bar{u}_h^i \end{bmatrix}^\top \begin{bmatrix} \nabla_x^2 c(\bar{x}_h^i, \bar{u}_h^i) & \nabla_{x,u}^2 c(\bar{x}_h^i, \bar{u}_h^i) \\ \nabla_{u,x}^2 c(\bar{x}_h^i, \bar{u}_h^i) & \nabla_u^2 c(\bar{x}_h^i, \bar{u}_h^i) \end{bmatrix} \begin{bmatrix} x - \bar{x}_h^i \\ u - \bar{u}_h^i \end{bmatrix}$$

$$+ \begin{bmatrix} x - \bar{x}_h^i \\ u - \bar{u}_h^i \end{bmatrix}^\top \begin{bmatrix} \nabla_x c(\bar{x}_h^i, \bar{u}_h^i) \\ \nabla_u c(\bar{x}_h^i, \bar{u}_h^i) \end{bmatrix} + c(\bar{x}_h^i, \bar{u}_h^i)$$

Formulate **time-dependent** LQR and compute its optimal control $\pi_0^i, \ldots, \pi_{H-1}^i$

Set new nominal trajectory: $\bar{x}_0^{i+1} = \bar{x}_0, \;\; \bar{u}_h^{i+1} = \pi_h^i(\bar{x}_h^{i+1}), \;$ and $\bar{x}_{h+1}^{i+1} = f(\bar{x}_h^{i+1}, \bar{u}_h^{i+1})$

Note this is true f, not approximation

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$$\min_{\alpha \in [0,1]} \sum_{h=0}^{H-1} c(x_h, \bar{u}_h^{i+1})$$

$$\text{s.t.} \quad x_{h+1} = f(x_h, \bar{u}_h^{i+1}), \quad \bar{u}_h^{i+1} = \alpha \bar{u}_h^i + (1-\alpha)\bar{u}_h, \quad x_0 = \bar{x}_0$$

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Why is this tractable?

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Why is this tractable? because it is 1-dimensional!

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Iterate between:

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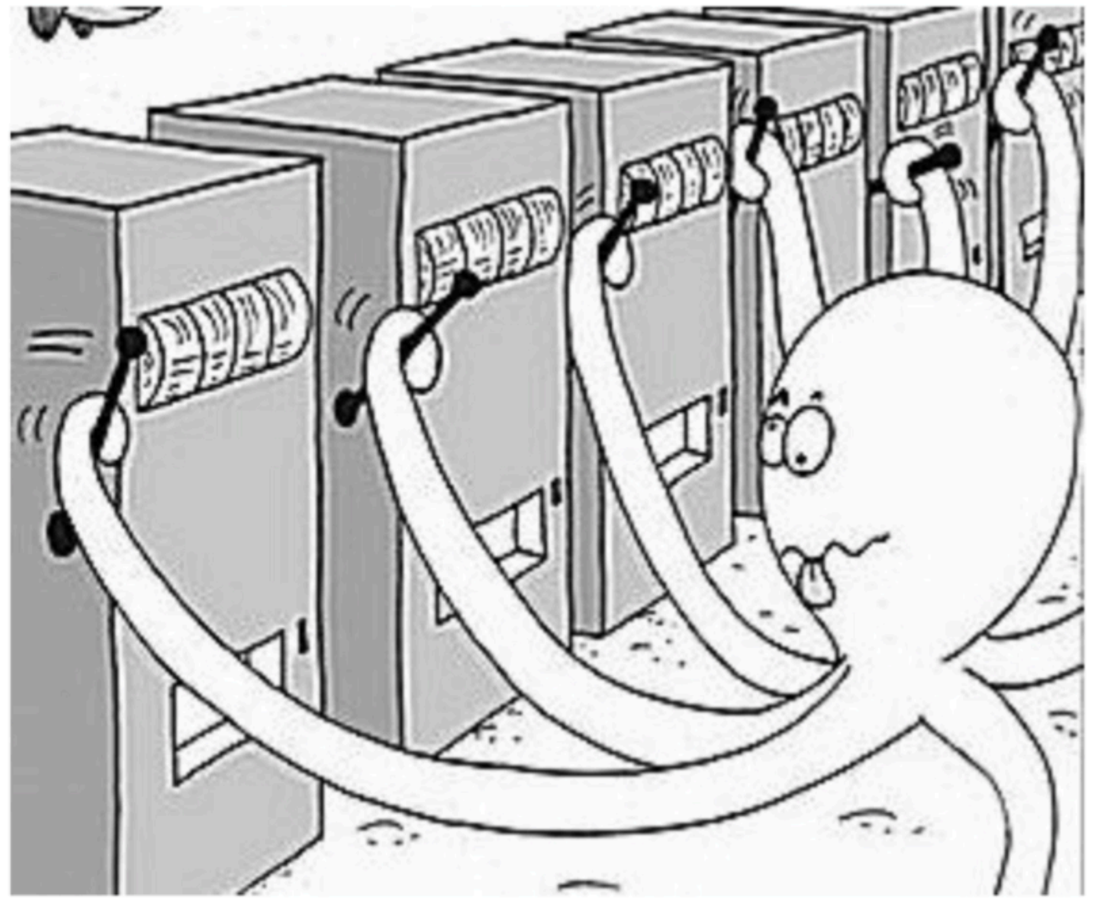
Intro to Multi-armed bandits (MAB)



Setting:

We have K many arms; label them $1, \ldots, K$

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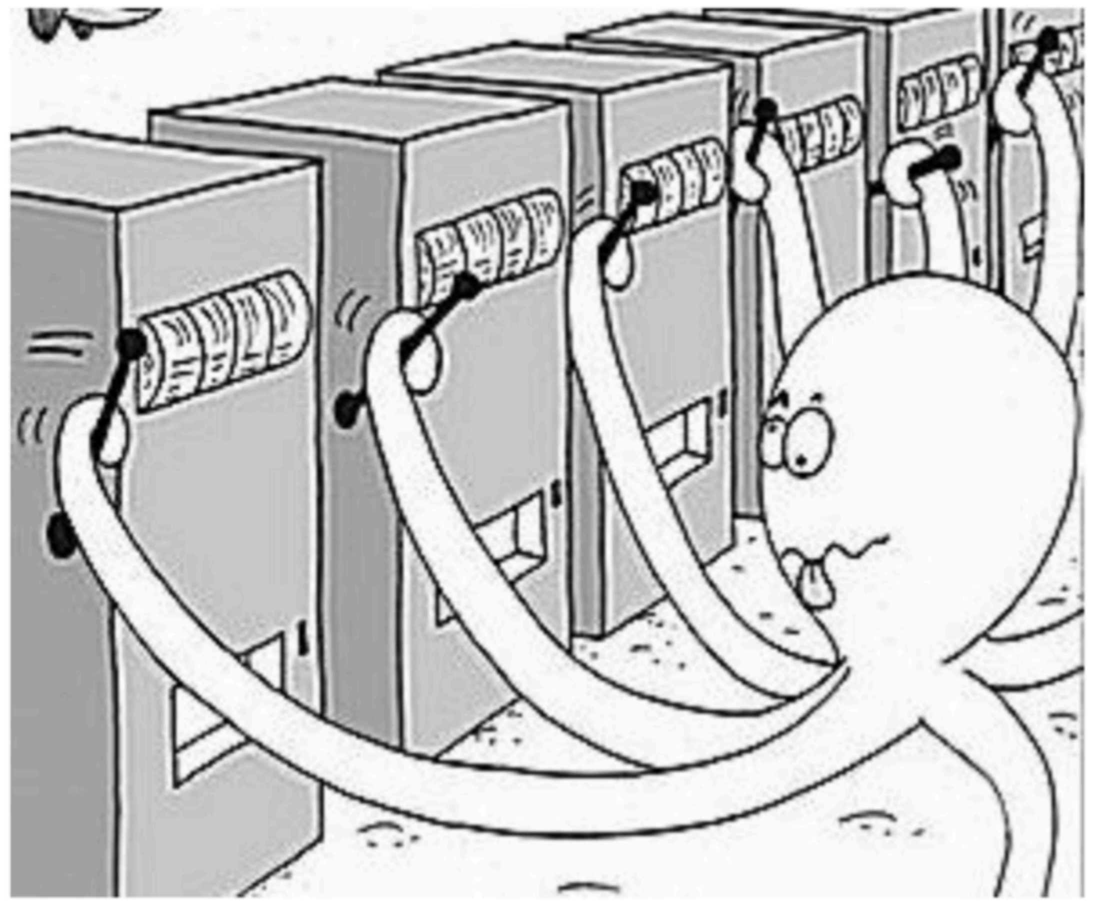

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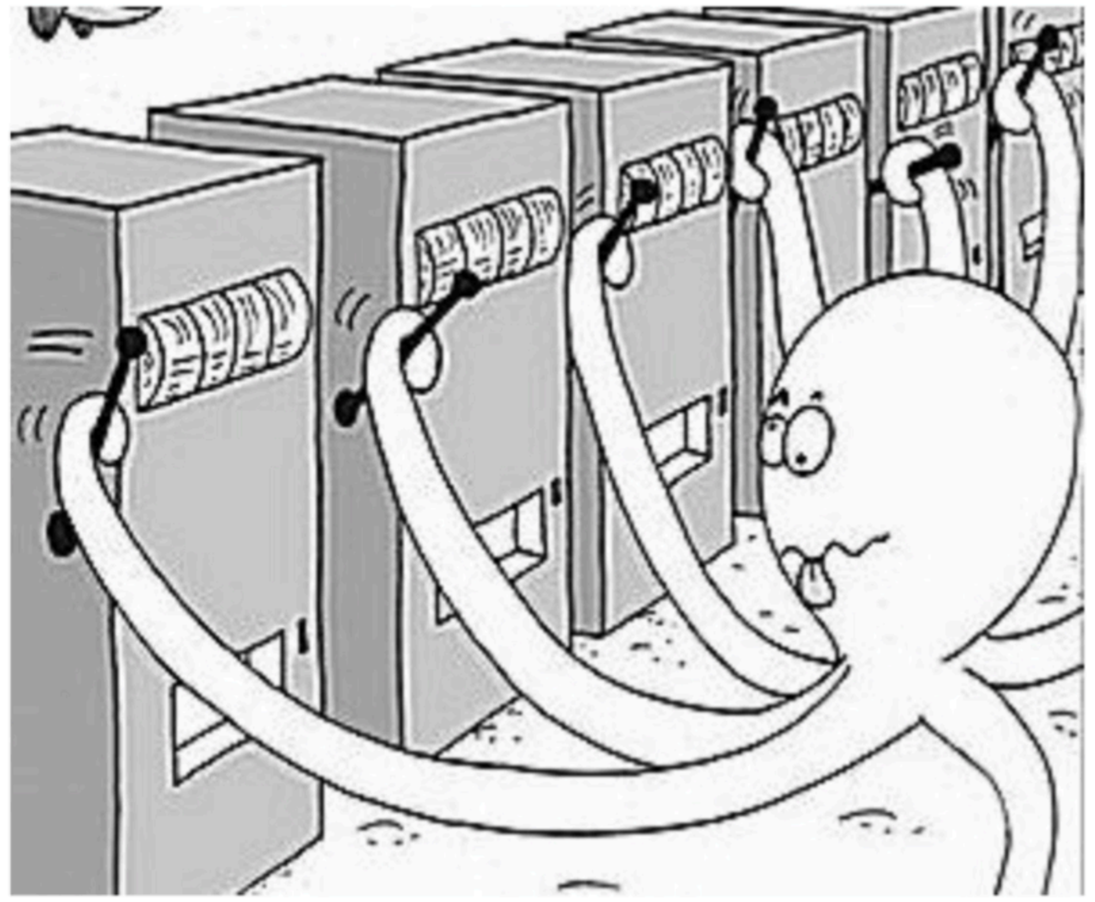

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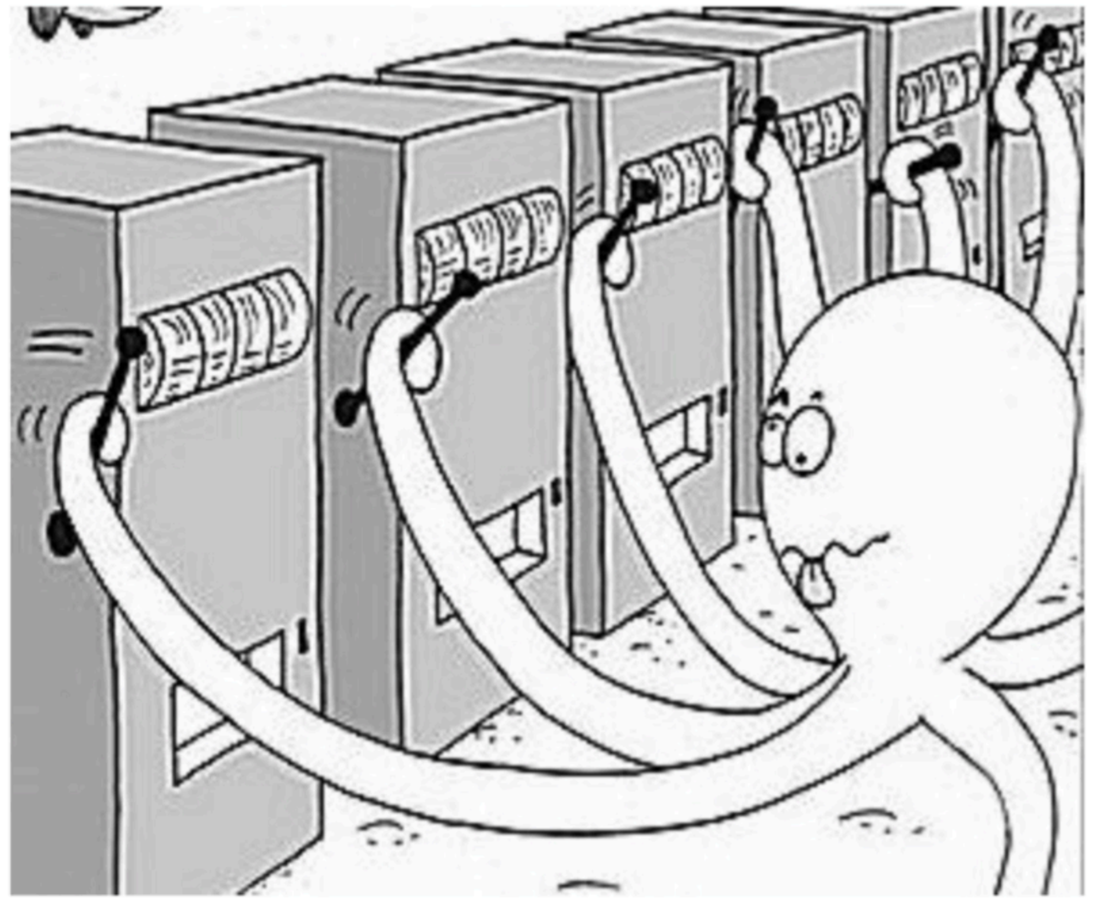

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Every time we pull arm k, we observe an i.i.d reward $r = \begin{cases} 1 & \text{w/ prob } \mu_k \\ 0 & \text{w/ prob } 1 - \mu_k \end{cases}$

Application: online advertising

Arms correspond to Ads

Reward is 1 if user clicks on ad

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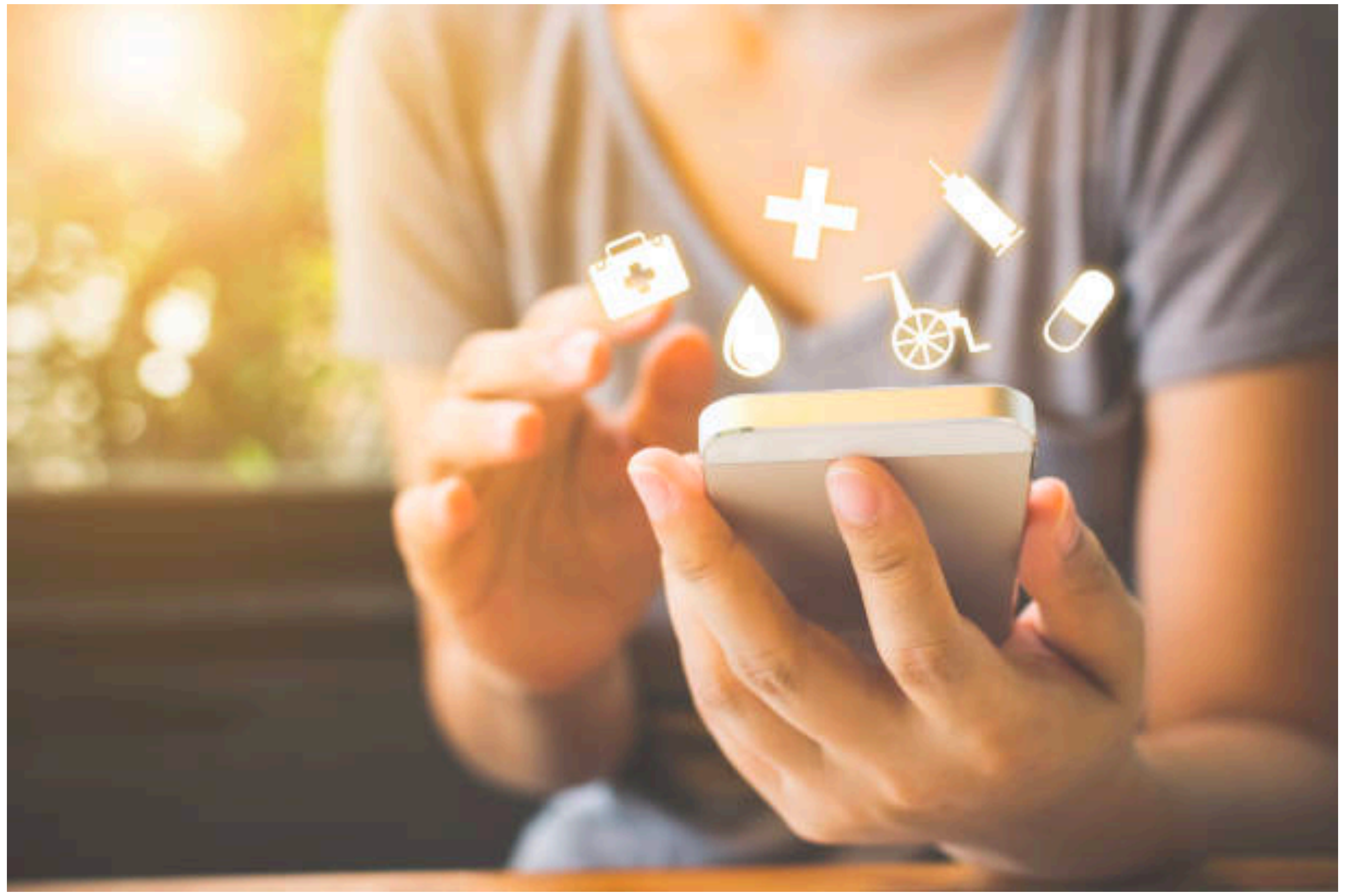
3. **Update**: Decide what ad to recommend for next round

Application: mobile health



Arms correspond to messages sent to users

Reward is, e.g., 1 if user exercised after seeing message

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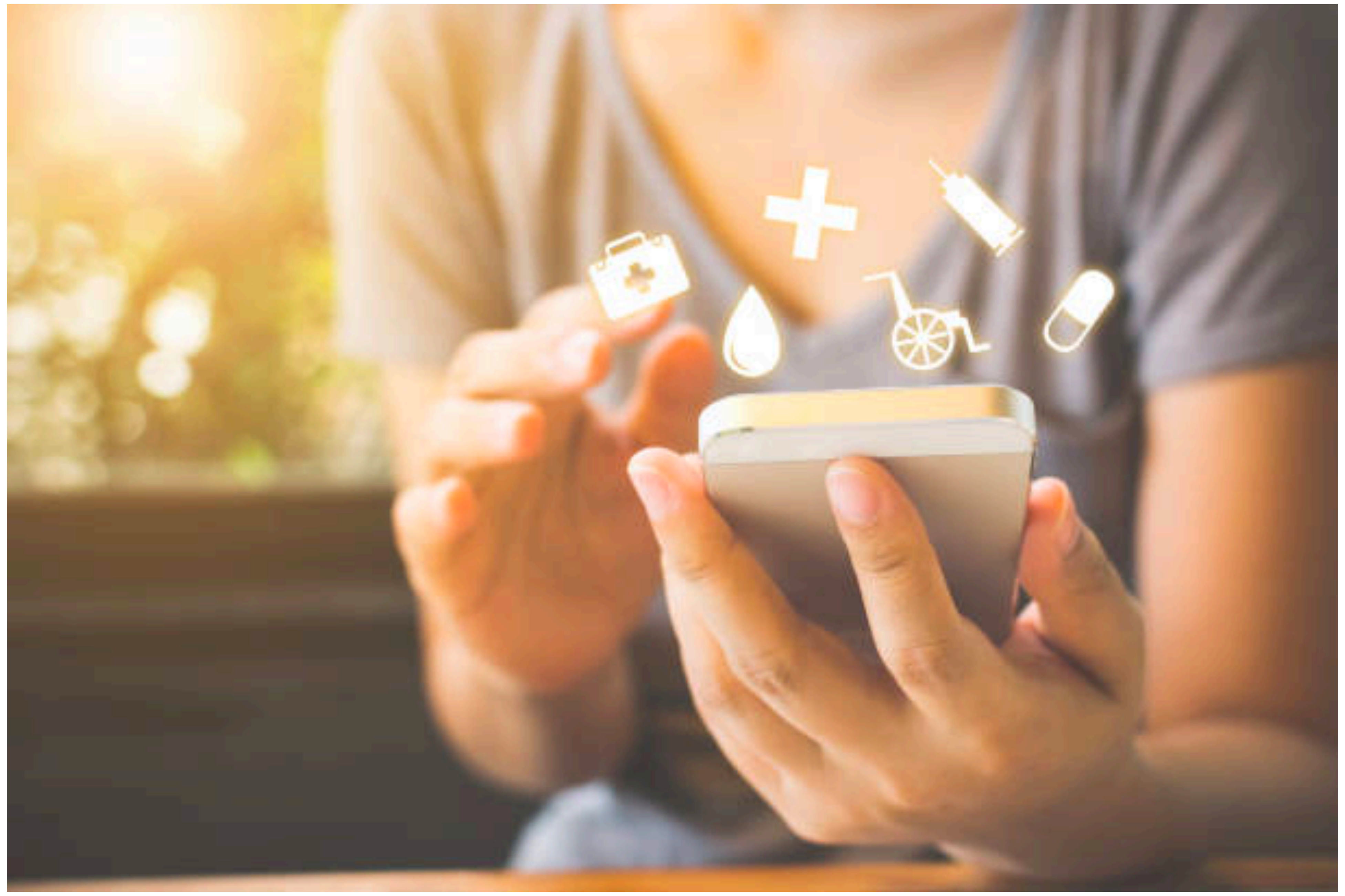

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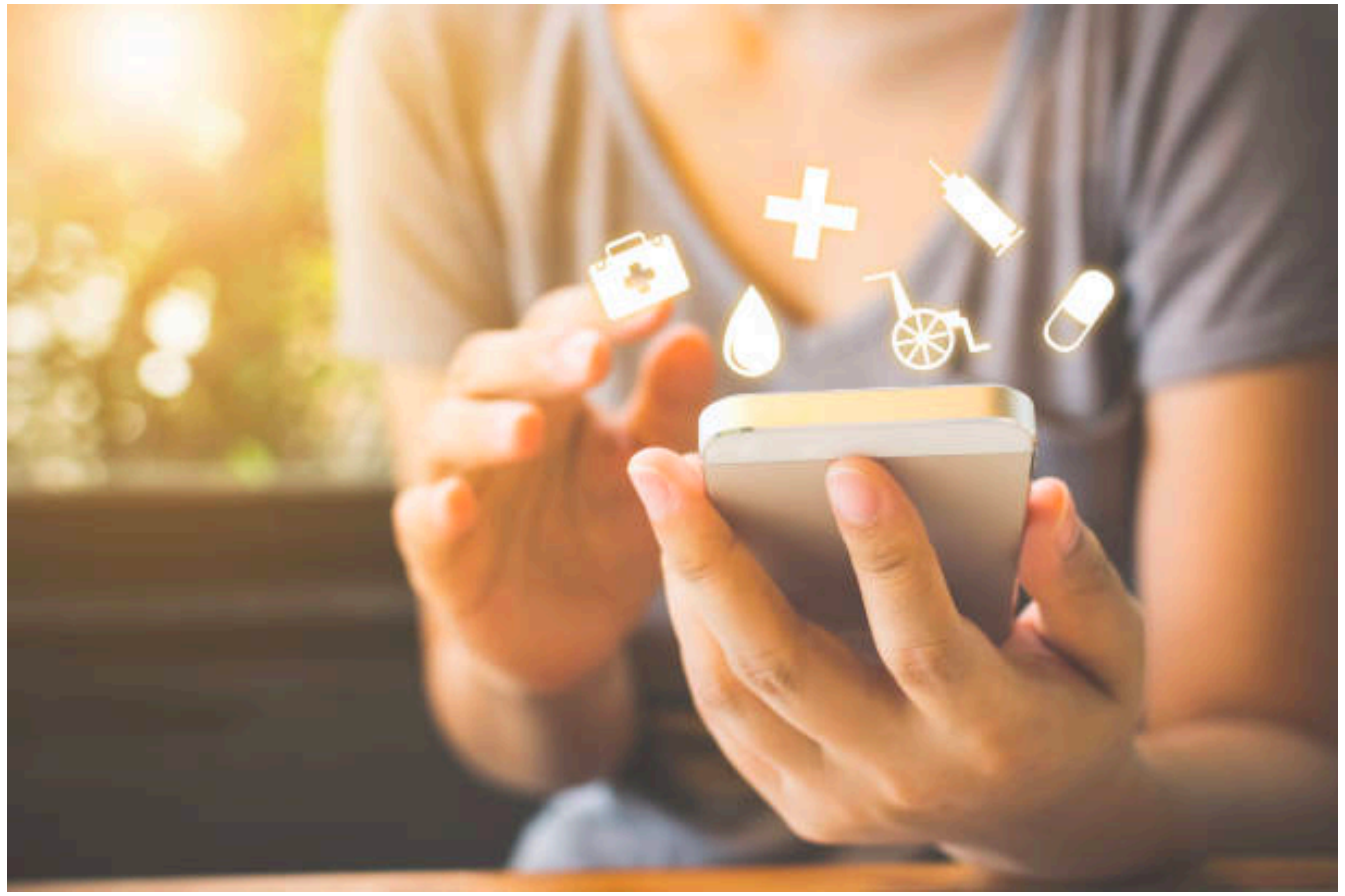

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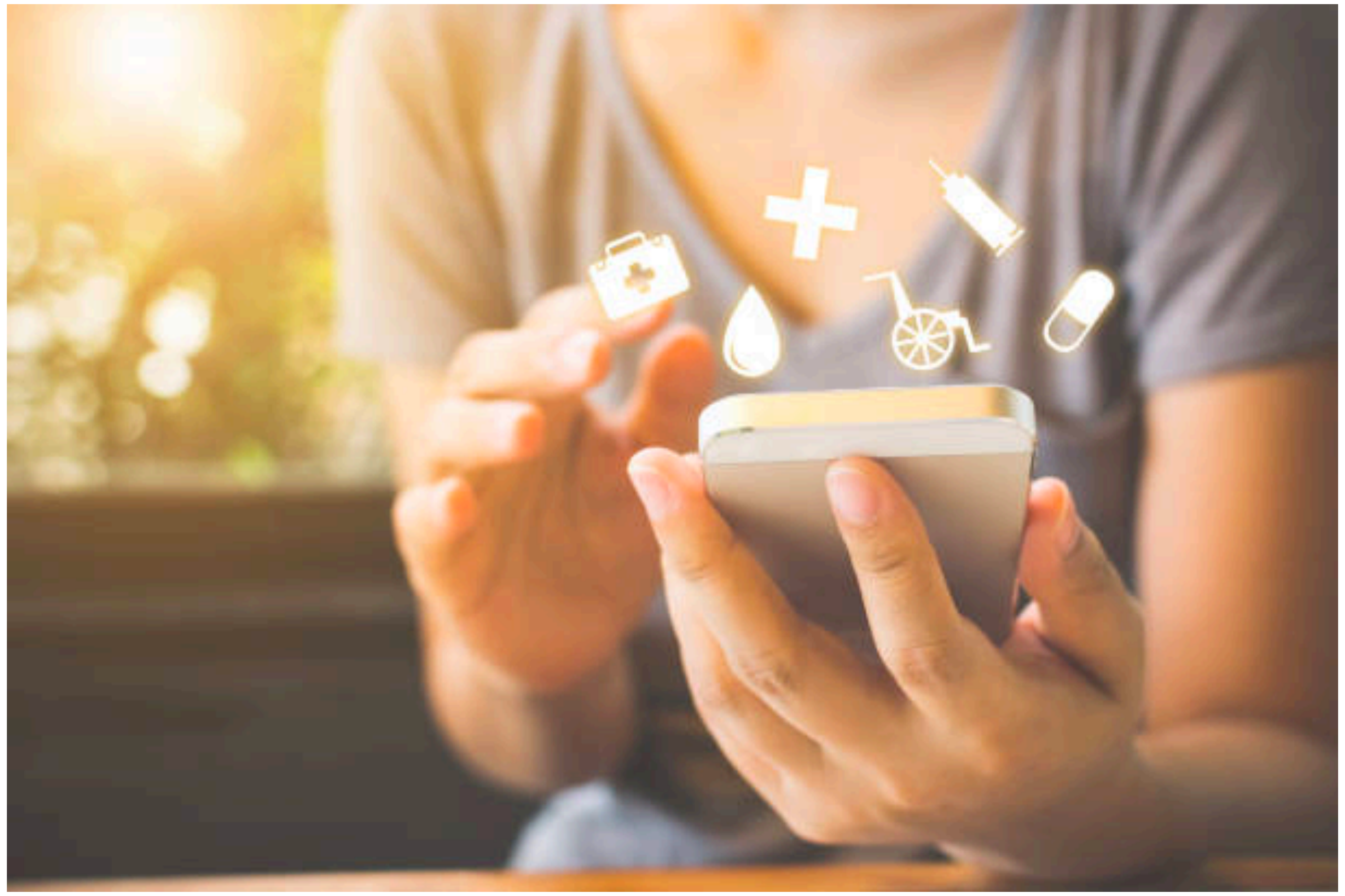

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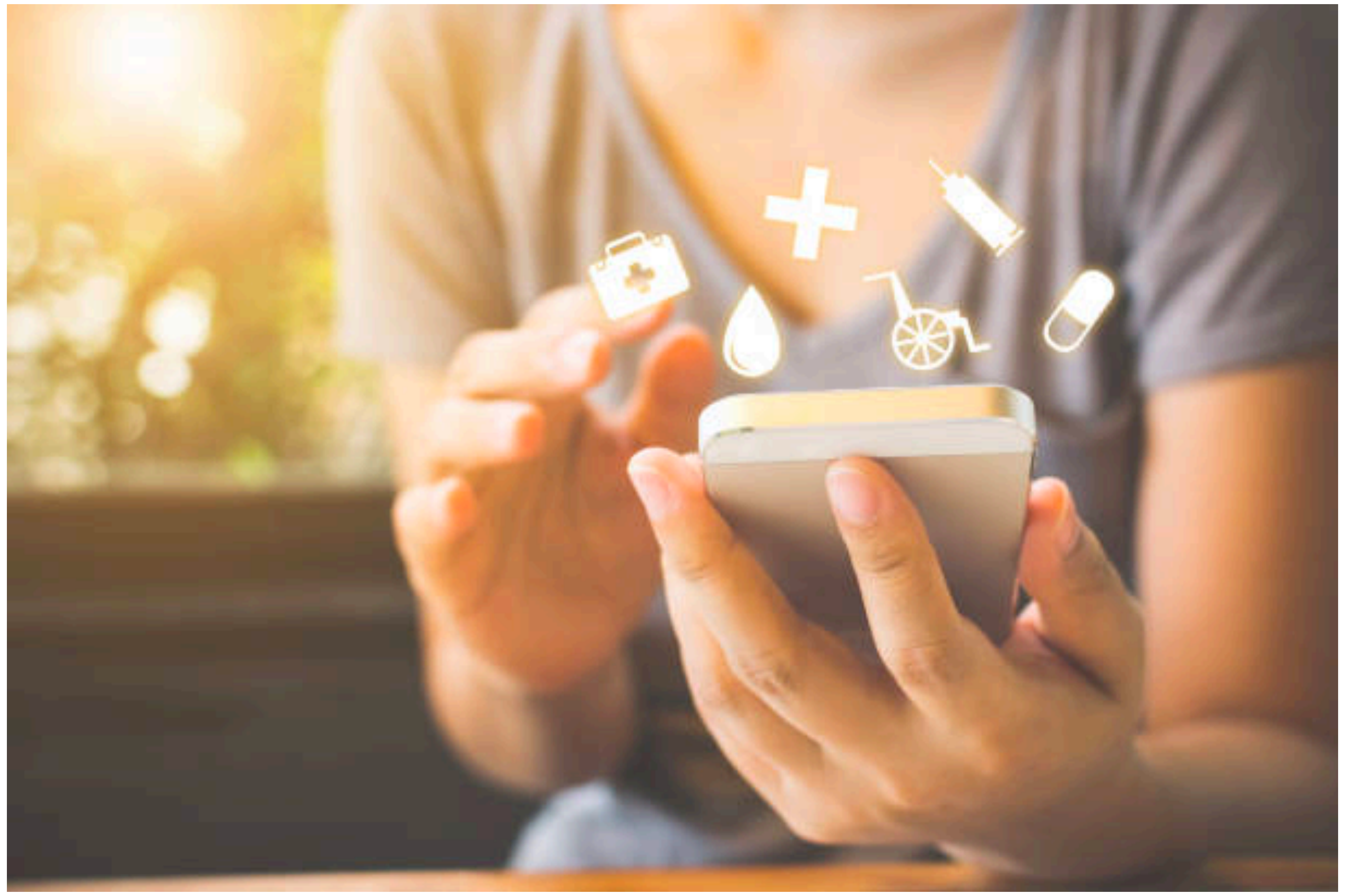

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More formally, we have the following interactive learning process:

For $t = 0 \to T - 1$

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Note: there is no state s; rewards from a given arm are i.i.d. (data NOT i.i.d.!)

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Why not sum the r_t?

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Exploration-Exploitation Tradeoff:

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Exploration-Exploitation Tradeoff:

Every round, we need to ask ourselves:

Should we pull the arm that currently appears best now (**exploit**; immediate payoff)?
Or pull another arm, in order to potentially learn it is better (**explore**; payoff later)?

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$$\mathbb{E}[\text{Regret}_T] = \mathbb{E}\left[T\mu^\star - \sum_{t=0}^{T-1} \mu_{a_t}\right] = T\left(\mu^\star - \bar{\mu}\right) = \Omega(T)$$

$$\bar{\mu} = \frac{1}{K}\sum_{k=1}^{K} \mu_k$$

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A bad arm (i.e., low μ_k) may generate a high reward by chance (or vice versa)!

Example: pure greedy

More concretely, let's say we have two arms:

Reward distribution for arm 1: ν_1 = Bernoulli($\mu_1 = 0.6$)

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Same rate as pure exploration!

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Lesson from pure greedy: exploring each arm once is not enough
Lesson from pure exploration: exploring each arm too much is bad too

Let’s allow both, and see how best to trade them off

Plan: (1) try each arm multiple times, (2) compute the empirical mean of each arm, (3) commit to the one that has the highest empirical mean

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For $k = 1,\ldots,K$: (Exploration phase)

Pull arm k N_e times to observe $\{r_i^{(k)}\}_{i=1}^{N_\mathsf{e}} \sim \nu_k$

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For $t = N_{\text{e}}K, \ldots, (T-1)$: (Exploitation phase)

Explore-Then-Commit (ETC)

N_{e} = Number of explorations

Algorithm hyper parameter $N_{\mathrm{e}} < T/K$ (we assume T >> K)

For $k = 1, \ldots, K$: (Exploration phase)

Pull arm k N_{e} times to observe $\{r_i^{(k)}\}_{i=1}^{N_{\mathrm{e}}} \sim \nu_k$

Calculate arm k's empirical mean: $\hat{\mu}_k = \frac{1}{N_{\mathrm{e}}} \sum_{i=1}^{N_{\mathrm{e}}} r_i^{(k)}$

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Q: how to set N_{e}?

Regret Analysis Strategy

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4. Minimize our upper-bound over N_{e}

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Hoeffding inequality

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Then with probability at least $1 - \delta$,

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- Why is this true? Full proof beyond course scope, but intuition easier…

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Hoeffding inequality: sample mean of N i.i.d. samples on $[0,1]$ satisfies

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- Don't worry too much about the extra 2's… CLT is only approximate!

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$$= \mu_{k^\star} + (\hat{\mu}_{k^\star} - \hat{\mu}_{k^\star}) - \mu_{\hat{k}} + (\hat{\mu}_{\hat{k}} - \hat{\mu}_{\hat{k}})$$

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$$\leq \sqrt{\ln(2K/\delta)/2N_{\mathsf{e}}} + \sqrt{\ln(2K/\delta)/2N_{\mathsf{e}}} + 0 \quad \text{w/p } 1 - \delta$$

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$$\begin{aligned}
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&= (\mu_{k^\star} - \hat{\mu}_{k^\star}) + (\hat{\mu}_{\hat{k}} - \mu_{\hat{k}}) + (\hat{\mu}_{k^\star} - \hat{\mu}_{\hat{k}}) \\
&\leq \sqrt{\ln(2K/\delta)/2N_{\mathsf{e}}} + \sqrt{\ln(2K/\delta)/2N_{\mathsf{e}}} + 0 \qquad \text{w/p } 1 - \delta \\
&= \sqrt{2\ln(2K/\delta)/N_{\mathsf{e}}}
\end{aligned}$$

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$$= \sqrt{2\ln(2K/\delta)/N_{\mathsf{e}}}$$

$$\Rightarrow \text{total regret during exploitation} \leq T\sqrt{2\ln(2K/\delta)/N_{\mathsf{e}}} \quad \text{w/p } 1 - \delta$$

Regret Analysis of ETC (cont’d)

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4. From steps 1-3: with probability $1 - \delta$,

$$\text{Regret}_T \leq N_{\text{e}} K + T\sqrt{2 \ln(2K/\delta)/N_{\text{e}}}$$

Regret Analysis of ETC (cont'd)

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$$\text{Regret}_T \leq N_\text{e} K + T\sqrt{2\ln(2K/\delta)/N_\text{e}}$$

Take any N_e so that $N_\text{e} \to \infty$ and $N_\text{e}/T \to 0$ (e.g., $N_\text{e} = \sqrt{T}$): <u>sub</u>linear regret!

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Minimize over N_e: (won't bore you with algebra)

$$\text{optimal } N_\text{e} = \left(\frac{T\sqrt{\ln(2K/\delta)/2}}{K} \right)^{2/3}$$

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(A bit more algebra to plug optimal N_e into Regret_T equation above)

$$\Rightarrow \text{Regret}_T \leq 3T^{2/3}(K\ln(2K/\delta)/2)^{1/3} = o(T)$$

Today

- ✓ Feedback from last lecture
- ✓ Recap
- ✓ Multi-armed bandit problem statement
- ✓ Baseline approaches: pure exploration and pure greedy
- ✓ Explore-then-commit

Summary:

- Multi-armed bandits (or MAB or just bandits)
  - Exemplify exploration vs exploitation
  - Pure greedy not much better than pure exploration (linear regret)
  - Explore then commit obtains sublinear regret

Attendance:

bit.ly/3RcTC9T

Feedback:

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