From LQR to Nonlinear Control

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CS/Stat 184(0): Introduction to Reinforcement Learning

Fall 2024

Today

- Feedback from last lecture
- Recap
- Locally linearization
- Iterative LQR

Feedback from feedback forms

1. Thank you to everyone who filled out the forms!
2. Positive definiteness of P_h at every step of induction

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Recap: LQR

Problem Statement (finite horizon, time homogeneous):

$$\arg \min_{\pi_0,\ldots,\pi_{H-1}:\mathbb{R}^d \to \mathbb{R}^k} \mathbb{E}\left[x_H^\top Q x_H + \sum_{h=0}^{H-1} (x_h^\top Q x_h + u_h^\top R u_h) \right]$$

such that $x_{h+1} = Ax_h + Bu_h + w_h\,, \quad x_0 \sim \mu_0\,, \quad u_h = \pi_h(x_h)\,, \quad w_h \sim N(0, \sigma^2 I)$

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- States $x_h \in \mathbb{R}^d$
- Actions/controls $u_h \in \mathbb{R}^k$
- Additive noise $w_h \sim \mathcal{N}(0,\sigma^2 I)$
- Dynamics linear with state coefficient matrix $A \in \mathbb{R}^{d\times d}$ and action coefficient matrix $B \in \mathbb{R}^{d\times k}$
- Cost function quadratic with positive definite state coefficient matrix $Q \in \mathbb{R}^{d\times d}$ and positive definite action coefficient matrix $R \in \mathbb{R}^{k\times k}$

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$$V_H^\star(x) = x^\top Q x, \text{ define } P_H = Q, p_H = 0,$$

We showed that $V_h^\star(x) = x^\top P_h x + p_h$, where:

$$P_h = Q + A^\top P_{h+1} A - A^\top P_{h+1} B (R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A$$

$$p_h = \text{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

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Along the way, we also showed that $\pi_h^\star(x) = -K_h x$, where:

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Optimal policy has nothing to do with initial distribution μ_0 or the noise σ^2!

Time-Dependent Costs and Dynamics

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such that $x_{h+1} = A_h x_h + B_h u_h + w_h\,, \quad x_0 \sim \mu_0\,, \quad u_h = \pi_h(x_h)\,, \quad w_h \sim N(0,\sigma^2 I)$

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Exact same derivation, only thing that changes is the Ricatti equation:

$$P_h = Q_h + A_h^\top P_{h+1} A_h - A_h^\top P_{h+1} B_h (R_h + B_h^\top P_{h+1} B_h)^{-1} B_h^\top P_{h+1} A_h$$

More General Quadratic Cost Function

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$$\arg \min_{\pi_0,\ldots,\pi_{H-1}:\mathbb{R}^d \to \mathbb{R}^k} \mathbb{E}\Big[x_H^\top Q_H x_H + x_H^\top q_H + c_H$$

$$+ \sum_{h=0}^{H-1} \left(x_h^\top Q_h x_h + u_h^\top R_h u_h + u_h^\top M_h x_h + x_h^\top q_h + u_h^\top r_h + c_h \right)\Big]$$

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Derivation is quite similar, just more algebra!

Tracking a Predefined Trajectory

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$$\arg \min_{\pi_0,\ldots,\pi_{H-1}:\mathbb{R}^d\to\mathbb{R}^k} \mathbb{E}\left[(x_H - x_H^\star)^\top Q_H (x_H - x_H^\star) + \sum_{h=0}^{H-1}\left((x_h - x_h^\star)^\top Q_h (x_h - x_h^\star) + (u_h - u_h^\star)^\top R_h (u_h - u_h^\star)\right)\right]$$

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Can you see why we already know how to solve this?

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Can you see why we already know how to solve this?

Expanding all the quadratic terms produces a special case of the previous slide!

Beyond LQR

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But what about problems with nonlinear dynamics and/or nonquadratic costs?

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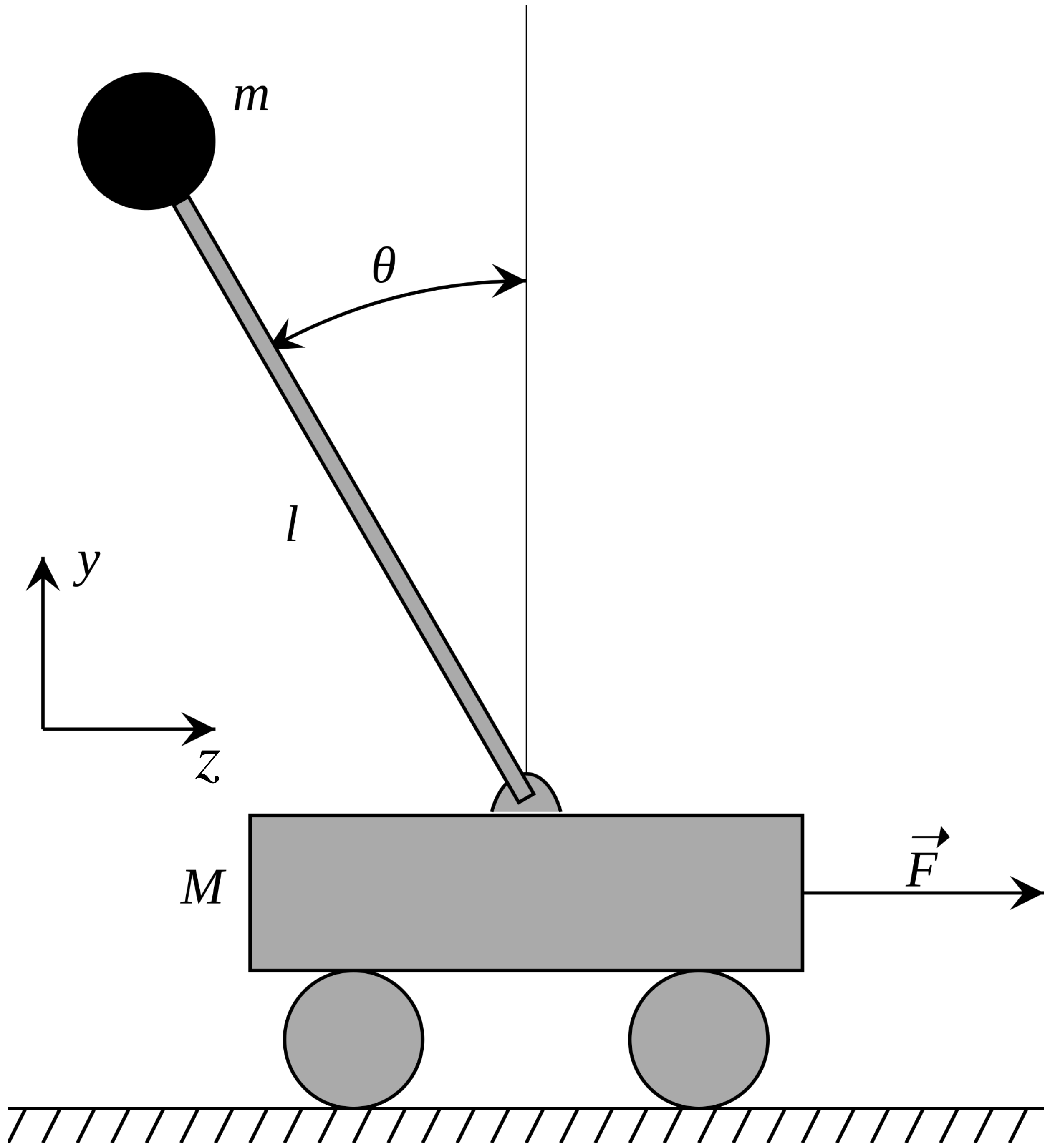
Setting for Local Linearization Approach:



Goal: stabilizing around the goal $(x = x^\star, u = u^\star)$

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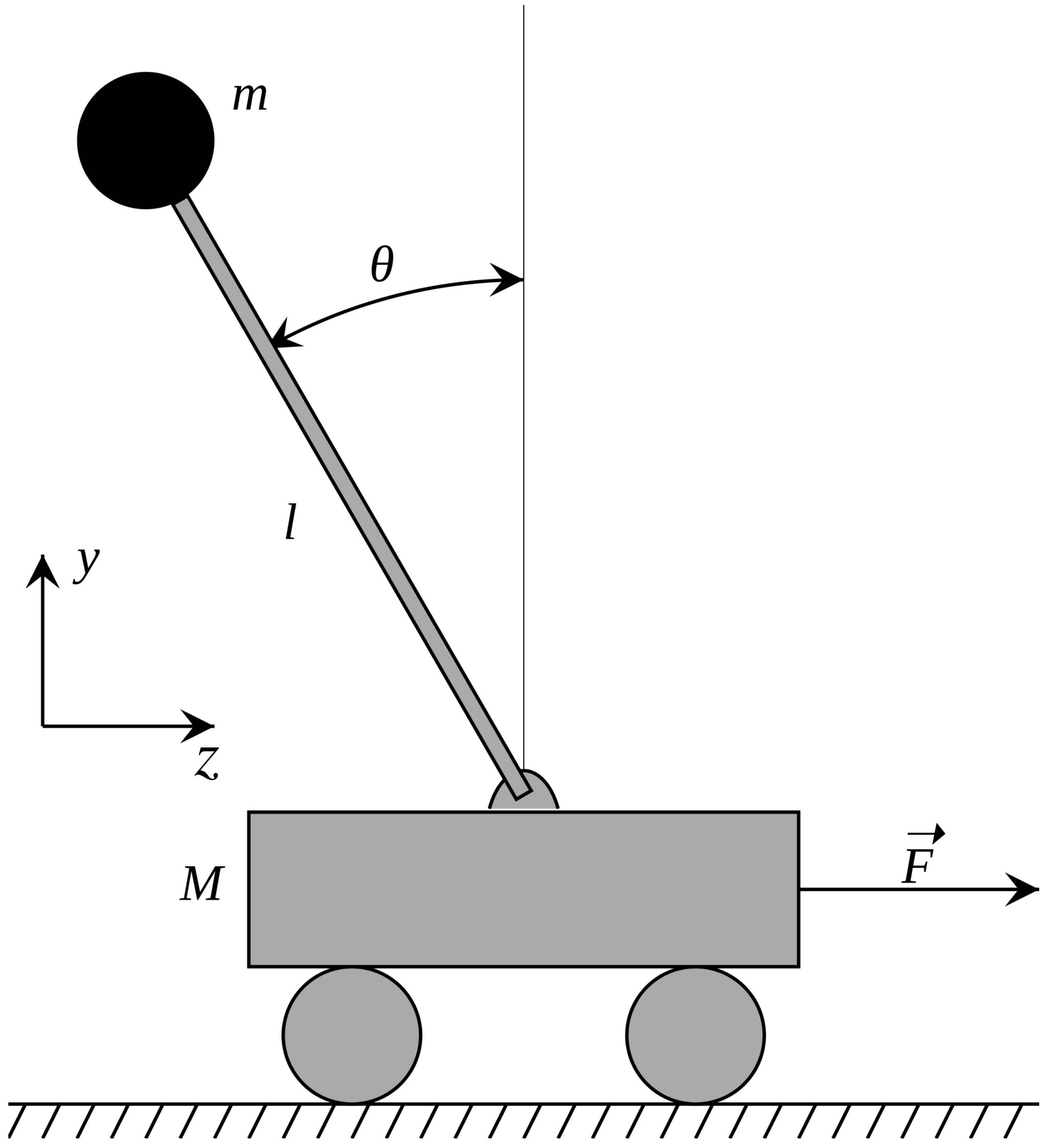

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$$\text{minimize } \mathbb{E}_\pi \Big[\sum_{h=0}^{H-1} c(x_h, u_h) \Big]$$

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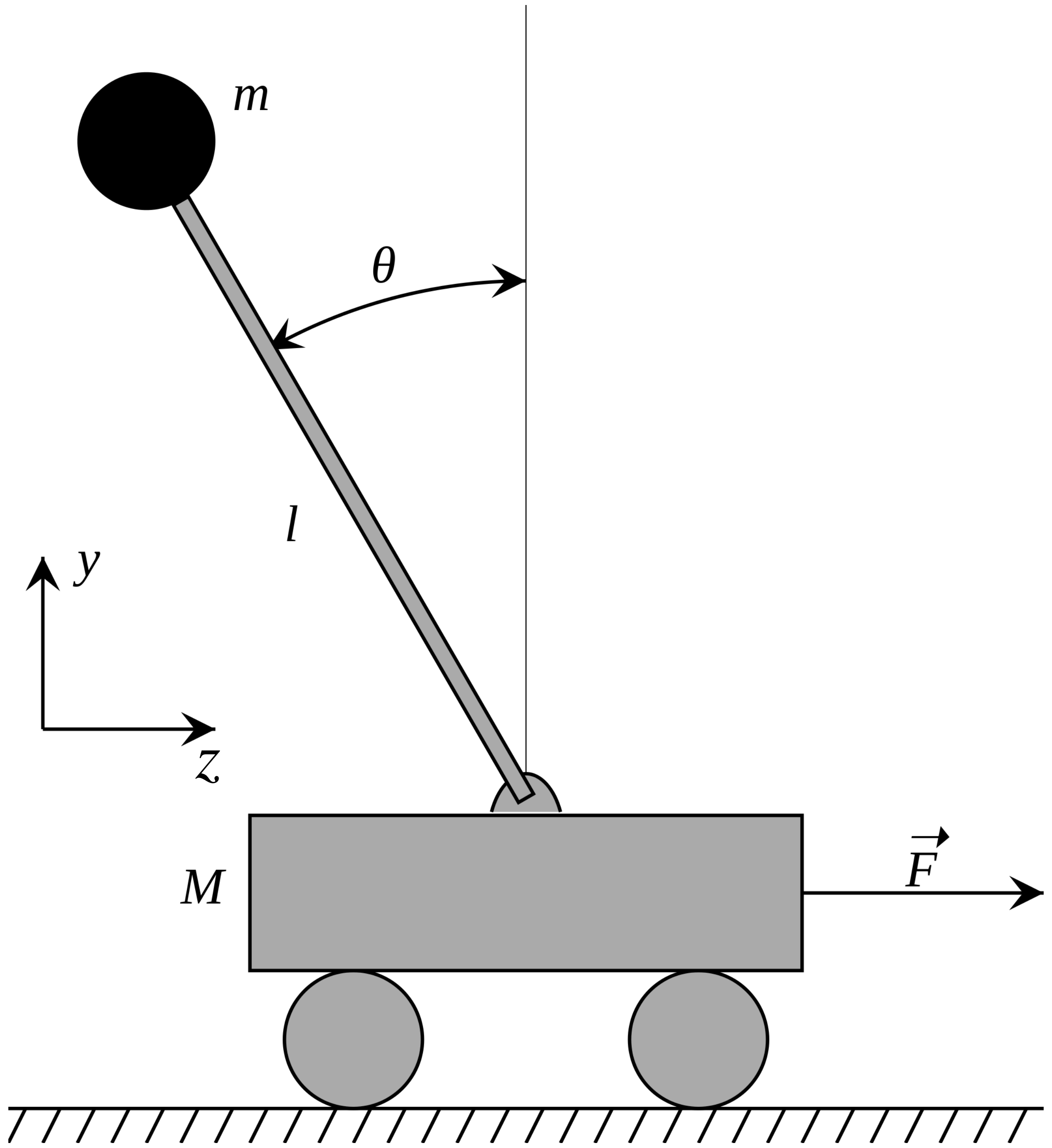

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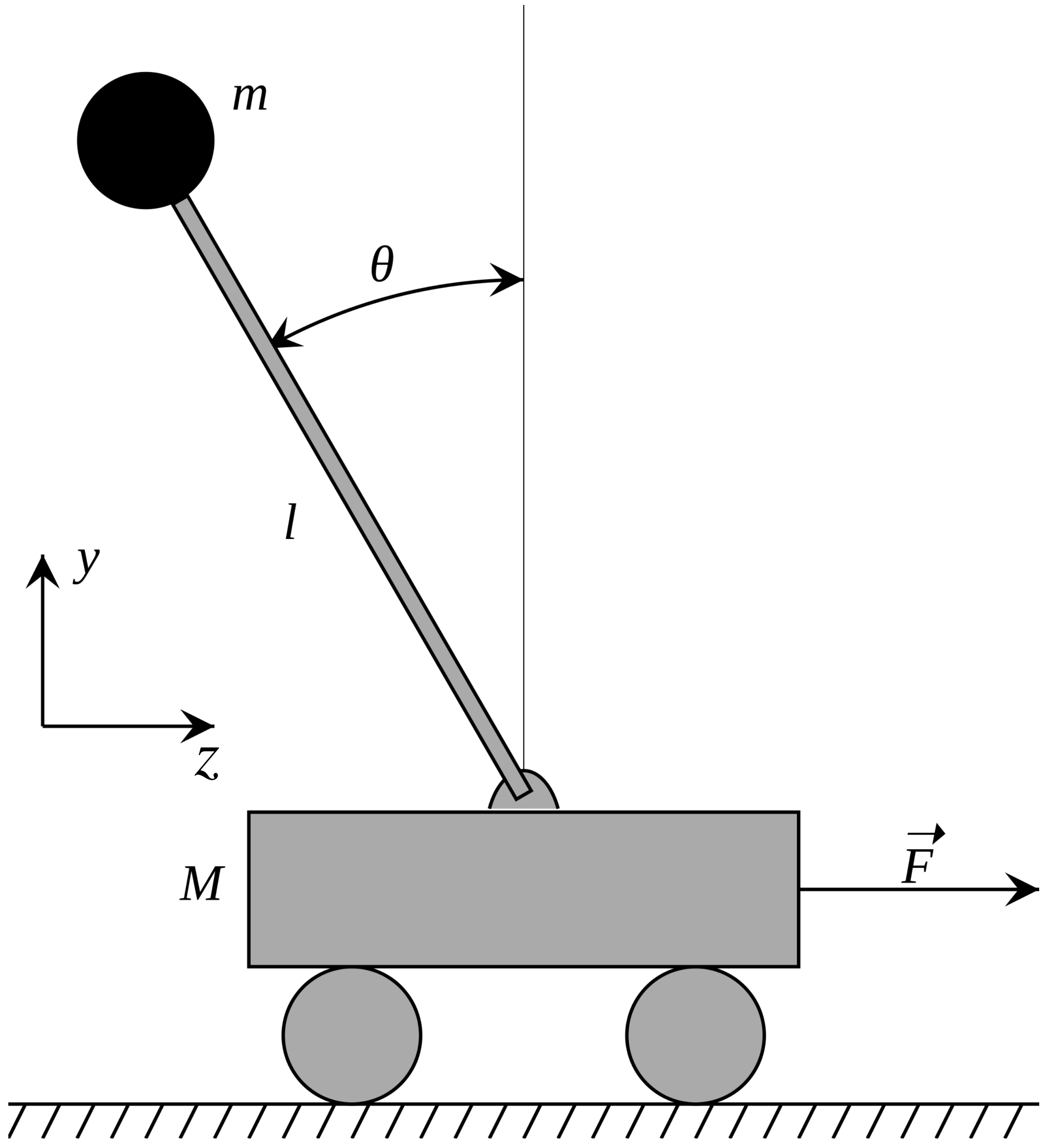

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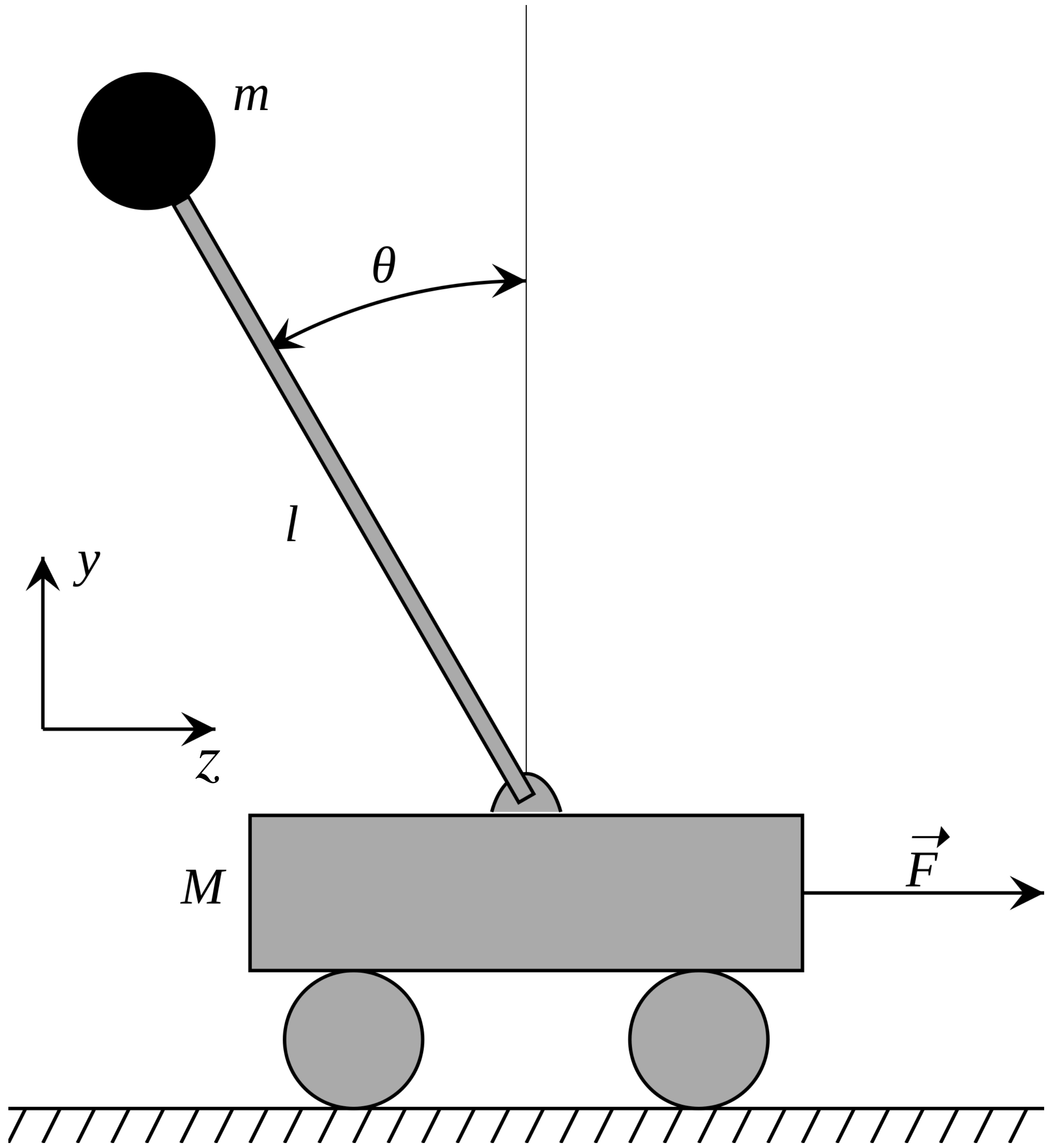

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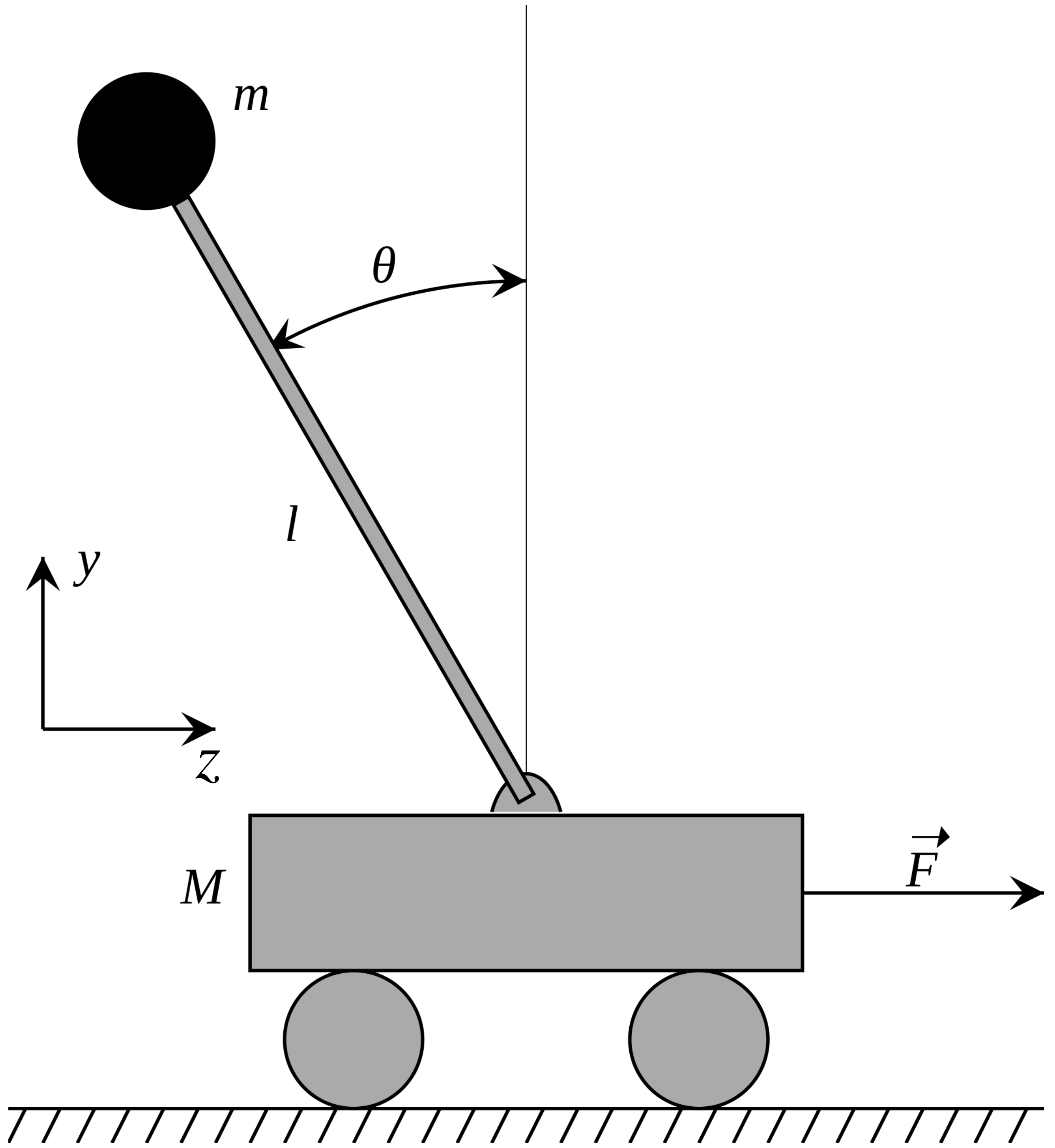

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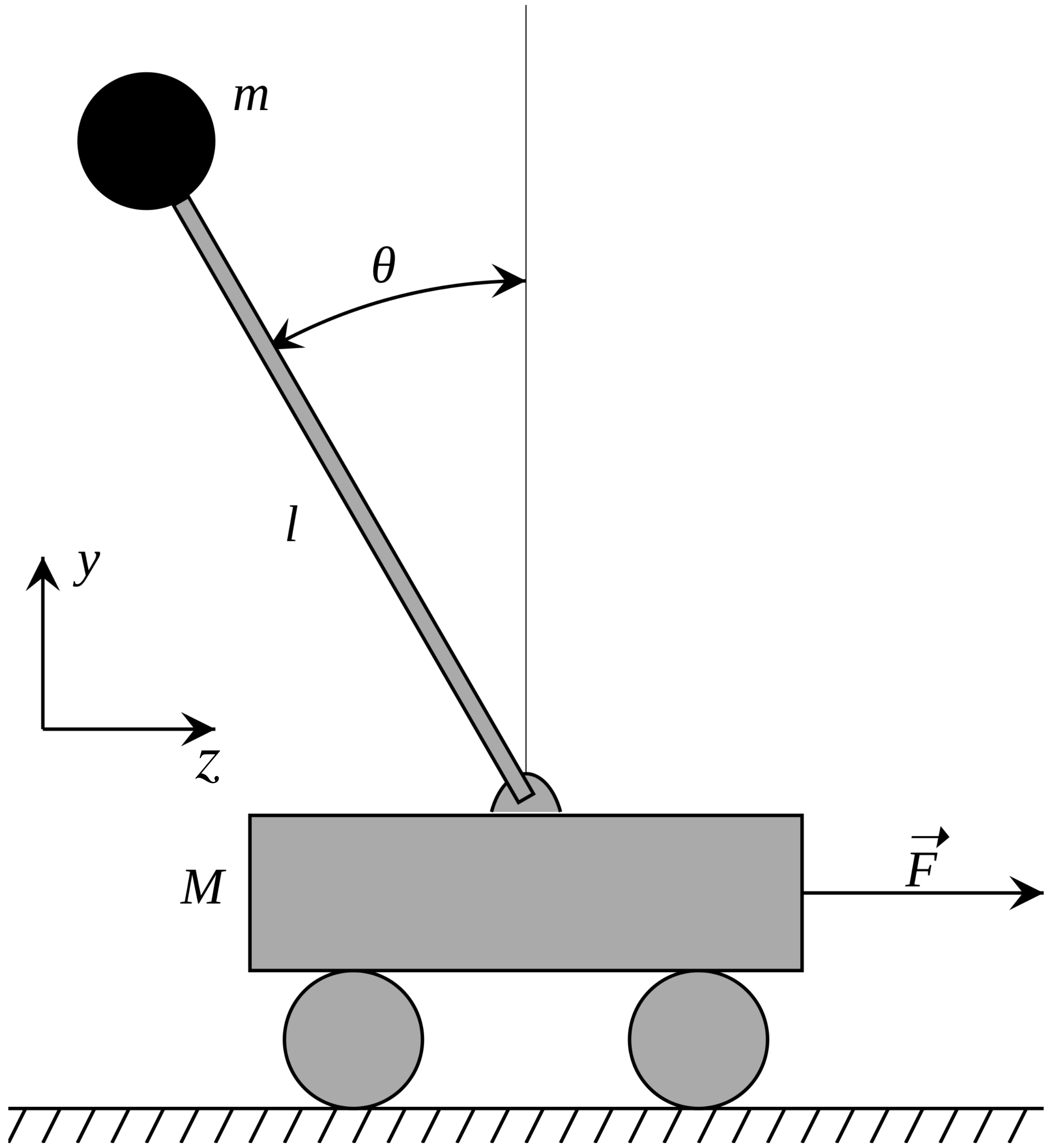

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2. f is differentiable and c is twice differentiable

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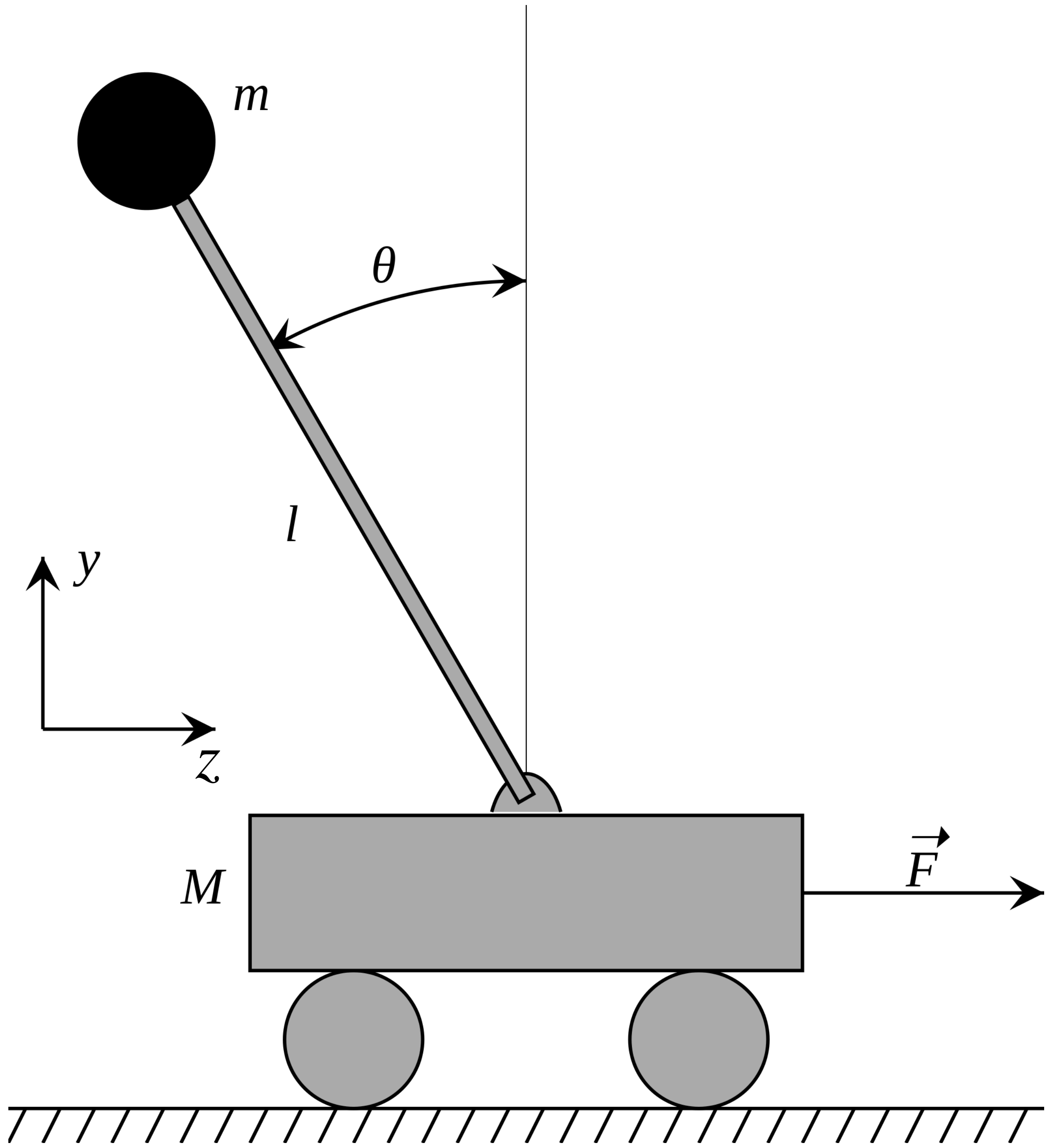

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$\nabla_x f(x, u), \nabla_u f(x, u), \nabla_x c(x, u), \nabla_u c(x, u),$
$\nabla_x^2 c(x, u), \nabla_u^2 c(x, u), \nabla_{x,u}^2 c(x, u)$

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We can approximate $f(x, u)$ locally with a first-order Taylor expansion:

$$f(x,u) \approx f(x^\star, u^\star) + \nabla_x f(x^\star, u^\star)\left(x - x^\star\right) + \nabla_u f(x^\star, u^\star)(u - u^\star)$$

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where:

$$\nabla_x f(x,u) \in \mathbb{R}^{d\times d}, \quad \nabla_x f(x,u)[i,j] = \frac{\partial f[i]}{\partial x[j]}(x,u)$$

$$\nabla_u f(x,u) \in \mathbb{R}^{d\times k}, \quad \nabla_u f(x,u)[i,j] = \frac{\partial f[i]}{\partial u[j]}(x,u)$$

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$$
\begin{aligned}
c(x,u) \approx\ & c(x^\star, u^\star) + \nabla_x c(x^\star, u^\star)^\top (x - x^\star) + \nabla_u c(x^\star, u^\star)^\top (u - u^\star) \\
& + \frac{1}{2}(x - x^\star)^\top \nabla_x^2 c(x^\star, u^\star)(x - x^\star) + \frac{1}{2}(u - u^\star)^\top \nabla_u^2 c(x^\star, u^\star)(u - u^\star) + (x - x^\star)^\top \nabla_{x,u}^2 c(x, u)(u - u^\star)
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$$

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$$\nabla_x c(x,u) \in \mathbb{R}^d, \quad \nabla_x c(x,u)[i] = \frac{\partial c}{\partial x[i]}(x,u),$$

$$\nabla_u c(x,u) \in \mathbb{R}^k, \quad \nabla_u c(x,u)[i] = \frac{\partial c}{\partial u[i]}(x,u),$$

$$\nabla_x^2 c(x,u) \in \mathbb{R}^{d\times d}, \quad \nabla_x^2 c(x,u)[i,j] = \frac{\partial^2 c}{\partial x[i]\partial x[j]}(x,u),$$

$$\nabla_{x,u}^2 c(x,u) \in \mathbb{R}^{d\times k}, \quad \nabla_{x,u}^2 c(x,u)[i,j] = \frac{\partial^2 c}{\partial x[i]\partial u[j]}(x,u)$$

Local Linearization: Putting it all Together

$$c(x,u) \approx c(x^\star,u^\star) + \nabla_x c(x^\star,u^\star)^\top (x - x^\star) + \nabla_u c(x^\star,u^\star)^\top (u - u^\star)$$
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$$f(x,u) \approx f(x^\star,u^\star) + \nabla_x f(x^\star,u^\star)\left(x - x^\star\right) + \nabla_u f(x^\star,u^\star)(u - u^\star)$$

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Rearranging terms, we get back to the following formulation:

$$\arg \min_{\pi_0,\ldots,\pi_{H-1}:\mathbb{R}^d \to \mathbb{R}^k} \quad \mathbb{E}\left[\sum_{h=0}^{H-1} (x_h^\top Q x_h + u_h^\top R u_h + u_h^\top M x_h + x_h^\top q + u_h^\top r + c)\right]$$

such that $\quad x_{h+1} = A x_h + B u_h + v\,, \quad x_0 \sim \mu_0\,, \quad u_h = \pi_h(x_h)$

Special case of one of the LQR extensions!

Summary of Local Linearization So Far:

For tasks such as balancing near goal state $(x^\star, u^\star)$,
we can perform **first order Taylor expansion on** $f(x, u)$,
and **second order Taylor expansion on** $c(x, u)$ around the balancing point $(x^\star, u^\star)$

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Last step: checking some practical issues

Locally Convexifying the Cost Function

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Note that $c(x, u)$ might not even be convex;

So, $\nabla_x^2 c(x^\star, u^\star)$ & $\nabla_u^2 c(x^\star, u^\star)$ may not be positive definite

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In practice, we force them to be positive definite:

Given a *symmetric matrix* $W \in \mathbb{R}^{d\times d}$,

we compute the eigen-decomposition $W = \sum_{i=1}^{d} \sigma_i z_i z_i^\top$, and we approximate W as

$$W \approx \sum_{i=1}^{d} \mathbf{1}(\sigma_i > 0)\sigma_i z_i z_i^\top + \lambda I,$$

for some small $\lambda > 0$

Computing Approximate Derivatives

Recall our assumption: we only have black-box access to f & c:

i.e., unknown analytical form, but given any (x, u), the black boxes output x', c, where

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$$\frac{\partial f[i]}{\partial x[j]}(x, u) \approx \frac{f(x + \delta_j, u)[i] - f(x - \delta_j, u)[i]}{2\delta}, \text{ where } \delta_j = [0, \ldots, 0, \underbrace{\delta}_{j'th \text{ entry}}, 0, \ldots 0]^\top$$

Computing Approximate Derivatives

Recall our assumption: we only have black-box access to f & c:

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To compute second derivative, e.g., $\frac{\partial^2 c}{\partial x[i] \partial u[j]}(x, u)$

First implement finite differencing procedure for $\partial c / \partial x[i]$, and then perform another finite differencing with respect to $u[j]$ on top of the first finite differencing procedure for $\partial c / \partial x[i]$

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4. The approximation is a (direct extension of) LQR, so we know how to compute the optimal policy

Today

- ✓ Feedback from last lecture
- ✓ Recap
- ✓ Locally linearization
- Iterative LQR

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But when x_h is far away from $x^\star$ or u_h needs to be far from $u^\star$ for any h, first/second-order Taylor expansion is not accurate anymore

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After linearization and quadratization at time h around waypoint $(\bar{x}_h, \bar{u}_h)$, $\forall h$, re-arranging terms gives:

$$\arg \min_{\pi_0, \ldots, \pi_{H-1}: \mathbb{R}^d \to \mathbb{R}^k} \mathbb{E}\left[\sum_{h=0}^{H-1} (x_h^\top Q_h x_h + u_h^\top R_h u_h + u_h^\top M_h x_h + x_h^\top q_h + u_h^\top r_h + c_h) \right]$$

such that $x_{h+1} = A_h x_h + B_h u_h + v_h\,, \quad x_0 \sim \mu_0\,, \quad u_h = \pi_h(x_h)$

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Time-dependent LQR problem: we know the solution

Question: how to choose the waypoints $(\bar{x}_h, \bar{u}_h)$ to get the best approximation/solution?

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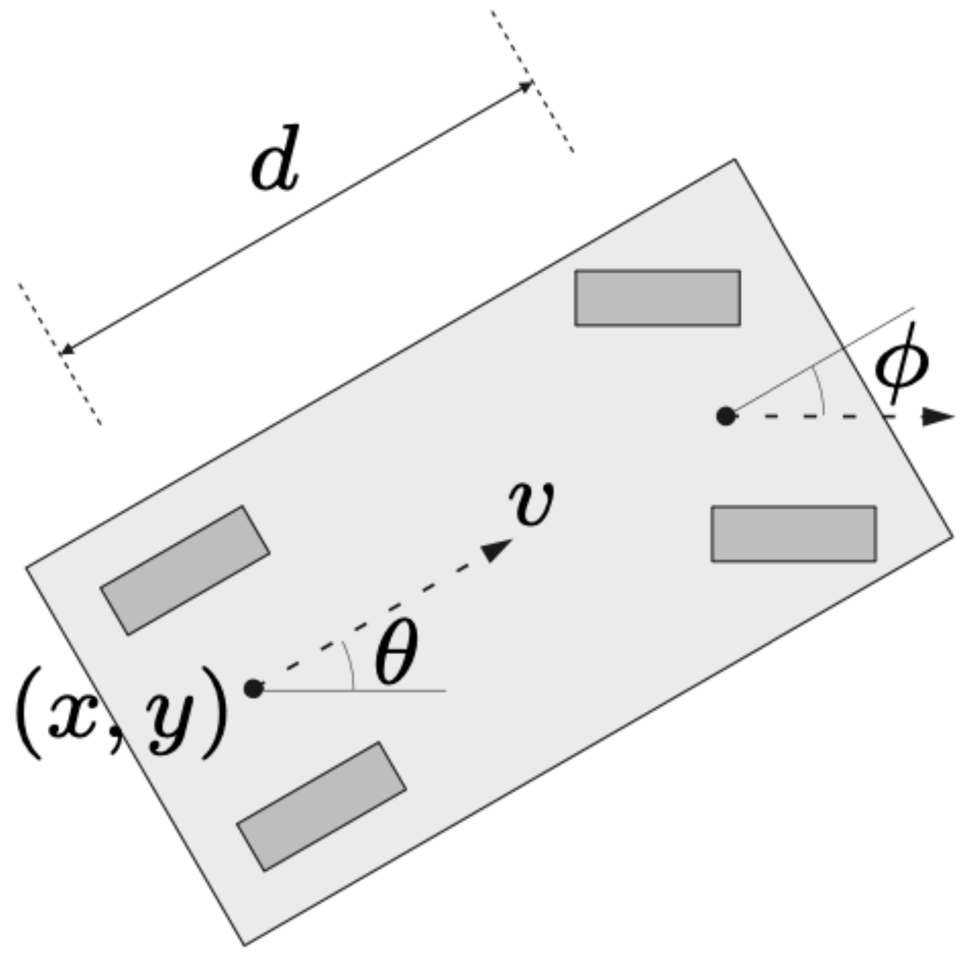
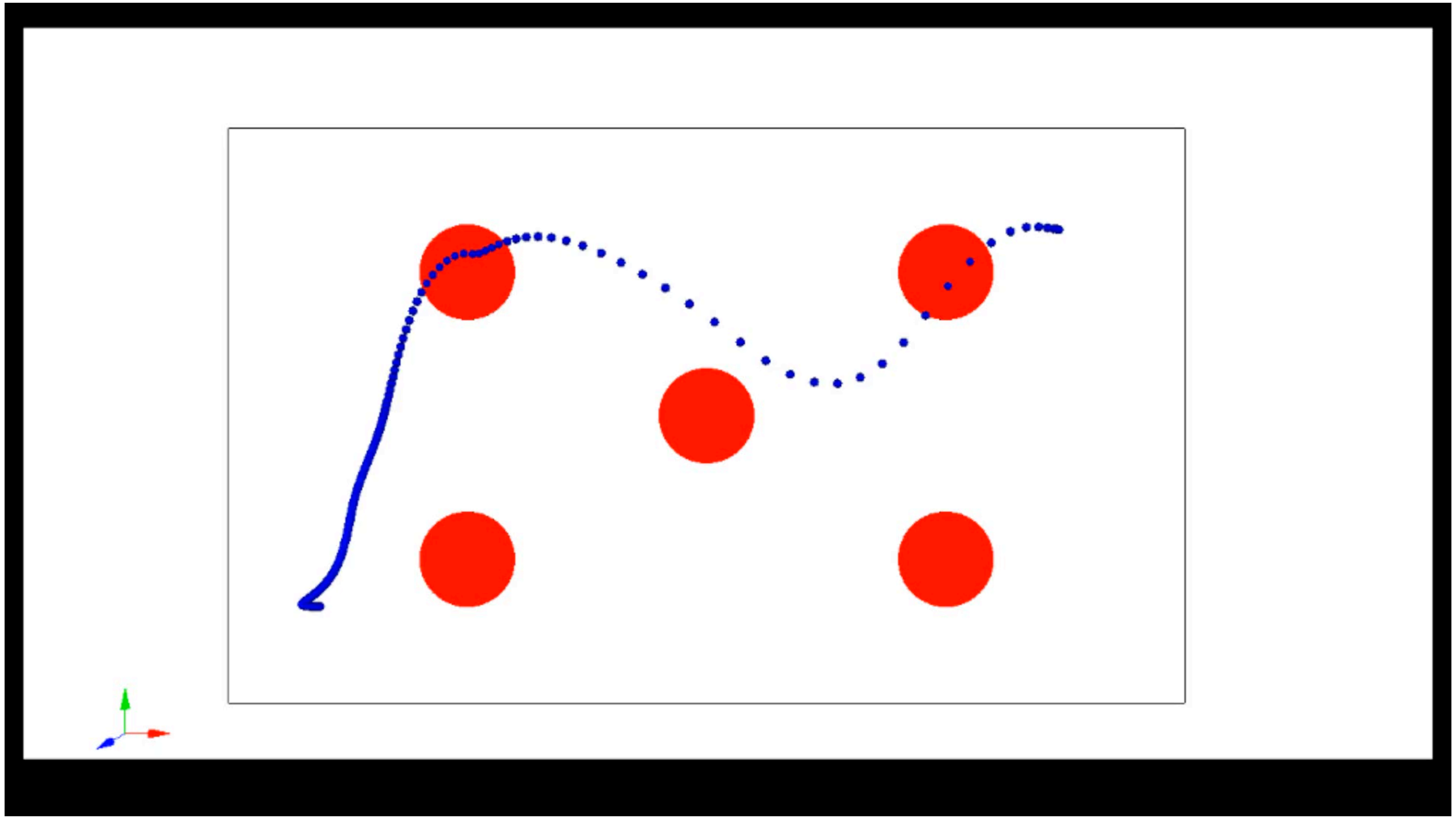
Why is this tractable? because it is 1-dimensional!

Example:

2-d car navigation

Cost function is designed such that it gets to the goal without colliding with obstacles (in red)

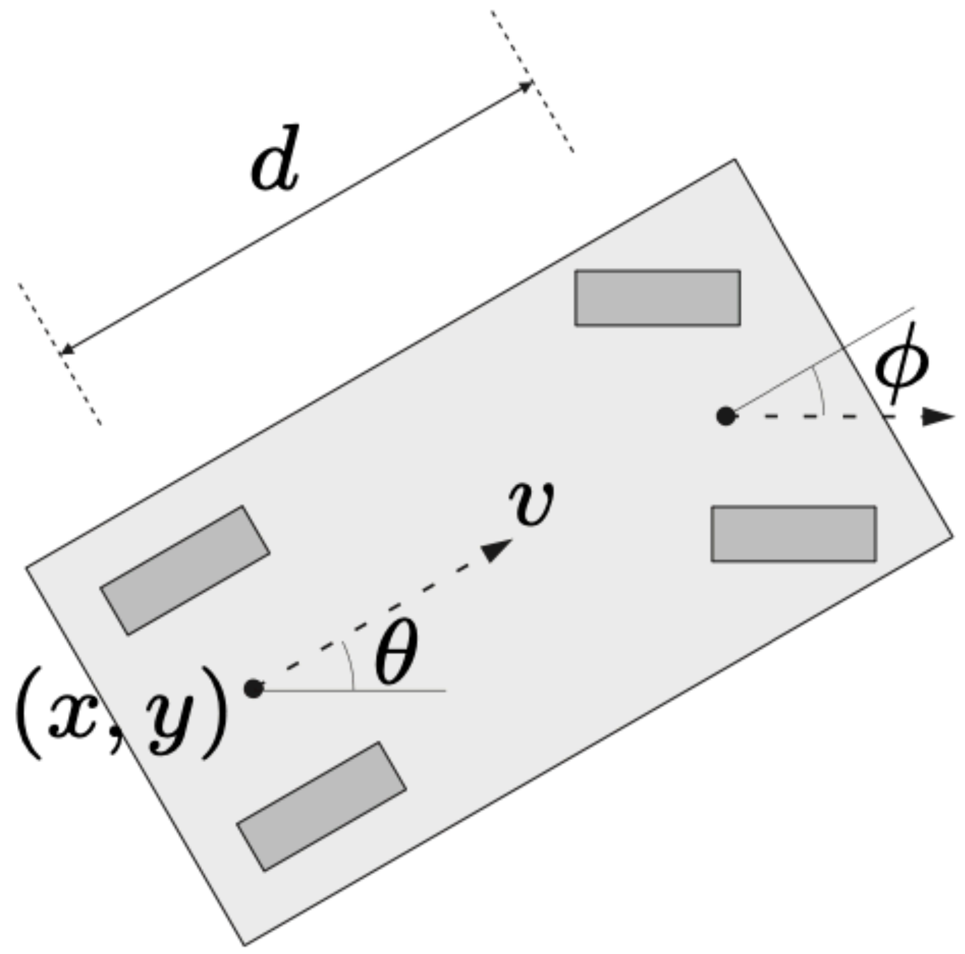


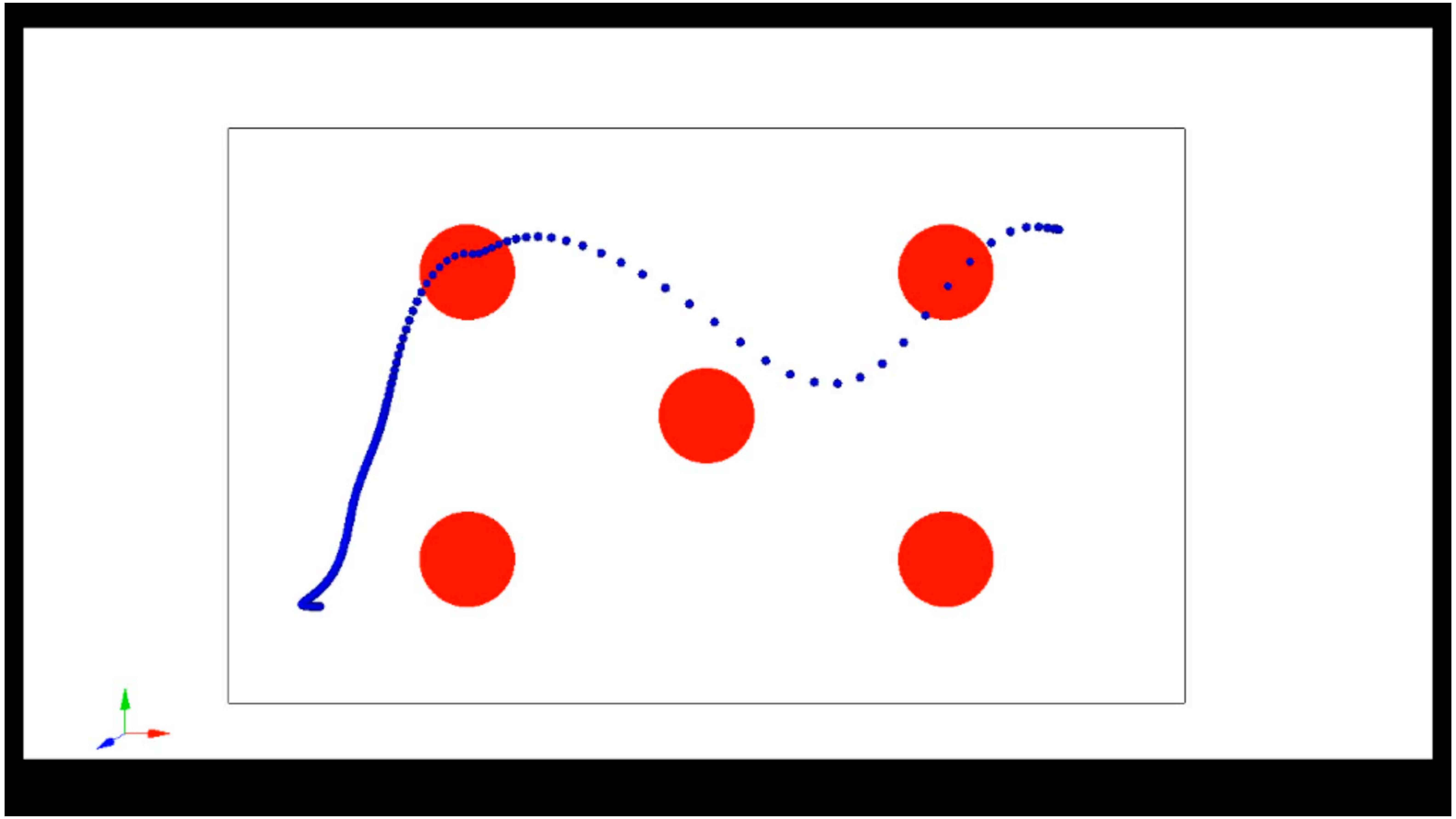


Example:

2-d car navigation

Cost function is designed such that it gets to the goal without colliding with obstacles (in red)





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Today

- Feedback from last lecture
- Recap
- Locally linearization
- Iterative LQR

Summary:

Local linearization

- Allows us to approximately optimally control any system near its optimum

Iterative LQR

- Uses LQR approximation to find locally optimal nonlinear control solution

Attendance:

bit.ly/3RcTC9T

Feedback:

bit.ly/3RHtlxy