Optimal Control Theory and the Linear Quadratic Regulator

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CS/Stat 184(0): Introduction to Reinforcement Learning

Fall 2024

Today

- Feedback from last lecture
- Recap
- General optimal control problem
- The linear quadratic regulator (LQR) problem
- Optimal control solution to LQR

Feedback from feedback forms

1. Thank you to everyone who filled out the forms!
2.

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Recap

Bellman Consistency and the Bellman Equations

- **Theorem:** Every policy π satisfies the Bellman consistency conditions:
  - $V^\pi(s) = r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,\pi(s))}[V^\pi(s')]$
- A function $V : S \to R$ satisfies the Bellman equations if

$$V(s) = \max_a \left\{ r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V(s') \right] \right\}, \forall s$$

- **Theorem:**
  - V satisfies the Bellman equations if and only if $V = V^\star$.

Value Iteration Algorithm:

1. Initialization: $V^0(s) = 0, \ \forall s$
2. For $t = 0, \ldots T-1$

   $$V^{t+1}(s) = \max_a \left\{ r(s,a) + \gamma \sum_{s' \in S} P(s' \mid s,a) V^t(s') \right\}, \ \forall s$$

3. Return: $V^T(s)$

   $$\pi(s) = \arg\max_a \left\{ r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot \mid s,a)} V^T(s') \right\}$$

- For $V \in \mathbb{R}^{|S|}$, define $\mathcal{T} : \mathbb{R}^{|S|} \mapsto \mathbb{R}^{|S|}$, where

  $$\left(\mathcal{T} V\right)(s) := \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V(s') \right]$$

- Bellman equations: $V = \mathcal{T} V$
- Value iteration: $V^{t+1} \leftarrow \mathcal{T} V^t$

Convergence of Value Iteration:

- The "infinity norm": For any vector $x \in R^d$, define $\|x\|_\infty = \max_i |x_i|$
- Theorem: Given any V, V', we have: $\|\mathcal{T}V - \mathcal{T}V'\|_\infty \leq \gamma \|V - V'\|_\infty$
- Corollary: If we set $T = \frac{1}{1-\gamma} \ln\left(\frac{1}{\epsilon(1-\gamma)}\right)$ iterations,

  VI will return a value V^T s.t. $\|V^T - V^\star\|_\infty \leq \epsilon$.

  - VI then has computational complexity $O(|S|^2|A|T)$.

Policy Iteration (PI)

- Initialization: choose a policy $\pi^0 : S \mapsto A$
- For $t = 0,1,\ldots T-1$
  1. **Policy Evaluation**: given π^t, compute $Q^{\pi^t}(s,a)$:
  2. **Policy Improvement**: set $\pi^{t+1}(s) := \arg\max_a Q^{\pi^t}(s,a)$

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- Computing Q^{π^t}
  - Computing V^{π^t}: $O(|S|^3)$ with linear system solving
  - Computing Q^{π^t} with V^{π^t}: $O(|S|^2|A|)$ using $Q^\pi(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[V^\pi(s')\right]$

Per iteration complexity: $O(|S|^3 + |S|^2|A|)$

Convergence of Policy Iteration:

- Theorem: PI has two properties:
  - montone improvement: $V^{\pi^{t+1}}(s) \geq V^{\pi^t}(s)$
  - "contraction": $\|V^{\pi^{t+1}} - V^\star\|_\infty \leq \gamma \|V^{\pi^t} - V^\star\|_\infty$

- Corollary: If we set $T = \frac{1}{1-\gamma} \ln\left(\frac{1}{\epsilon(1-\gamma)}\right)$ iterations,

  PI will return a policy π^{t+1} s.t. $\|V^{\pi^{t+1}} - V^\star\|_\infty \leq \epsilon$

  - with total computational complexity $O\Big(\big(|S|^3 + |S|^2 |A| \big) T \Big)$.

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- But all algorithms scale polynomially in the size of the state and action spaces… what if one or both are infinite?
- In this unit (next 2 lectures), we will discuss computation of good/optimal policies in continuous/infinite state and action spaces

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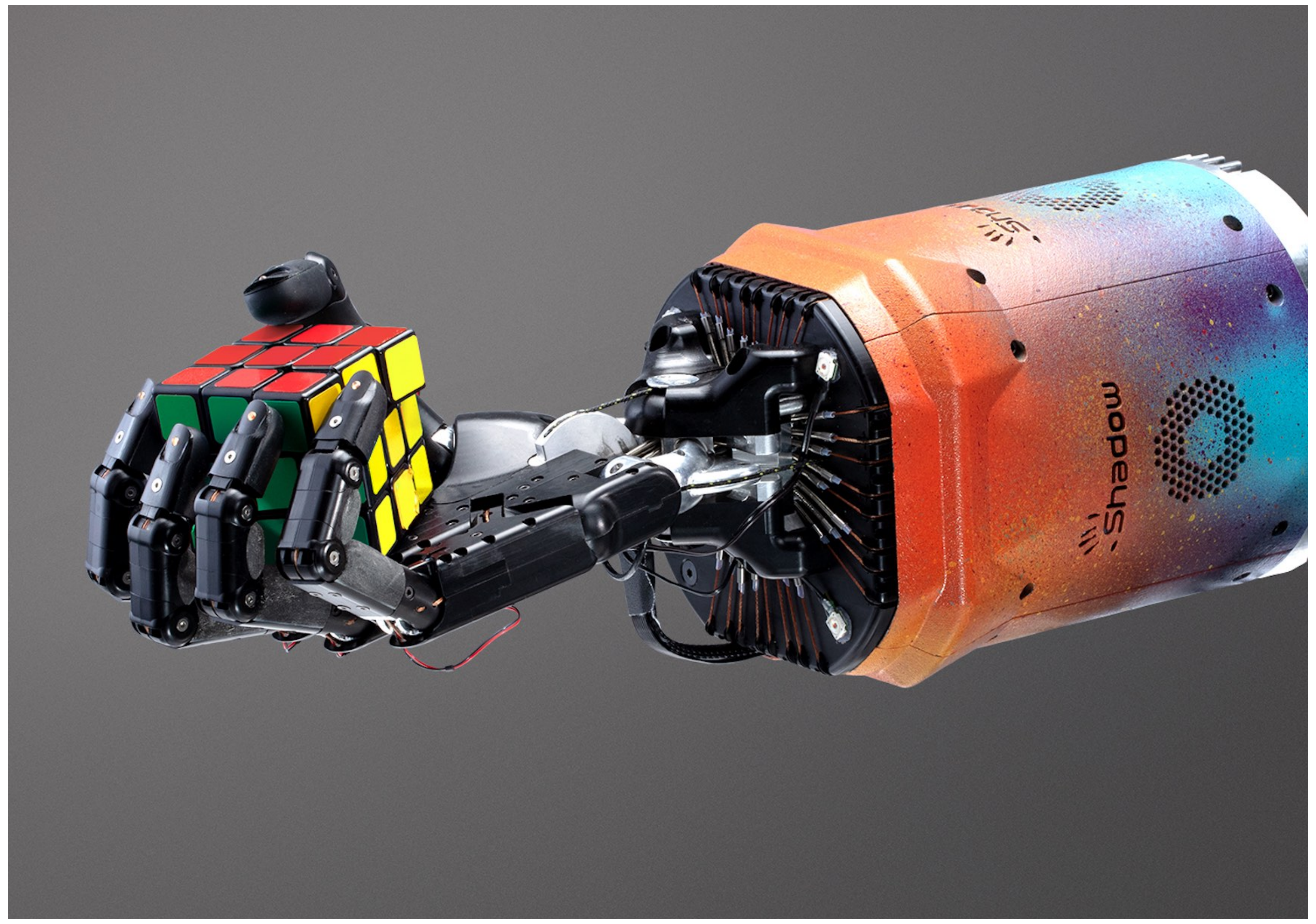
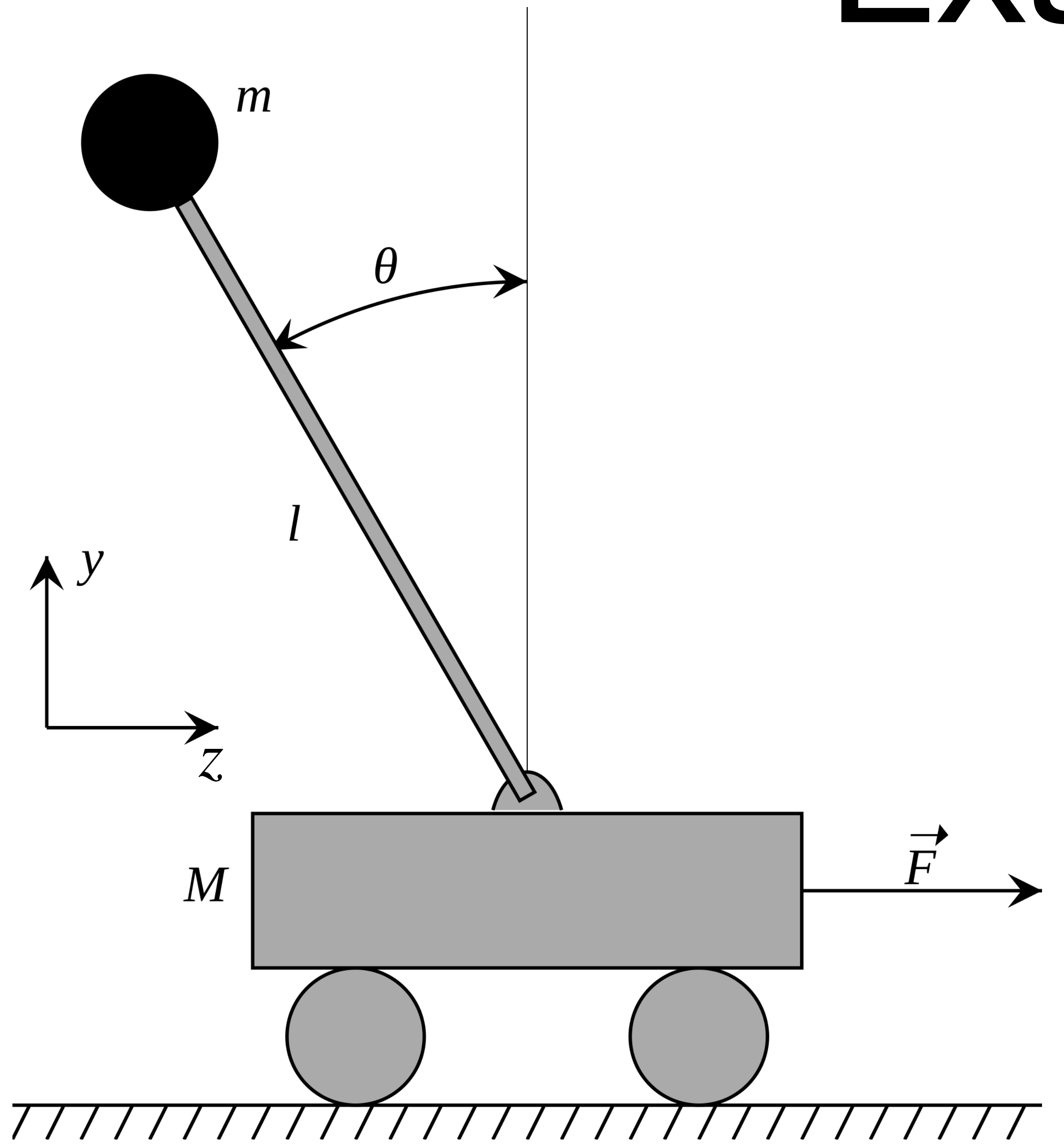
Robotics and Controls



Example: CartPole



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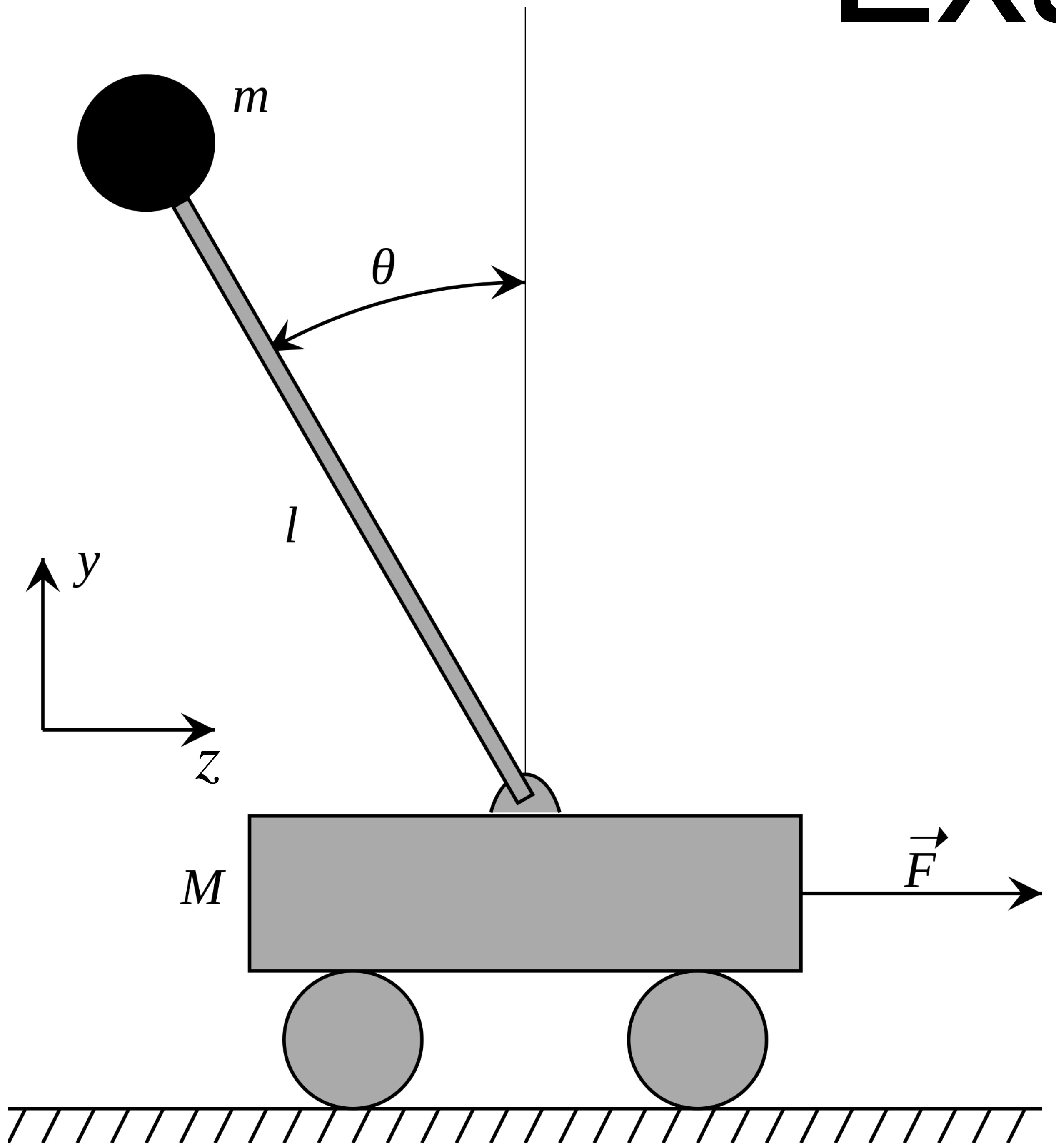

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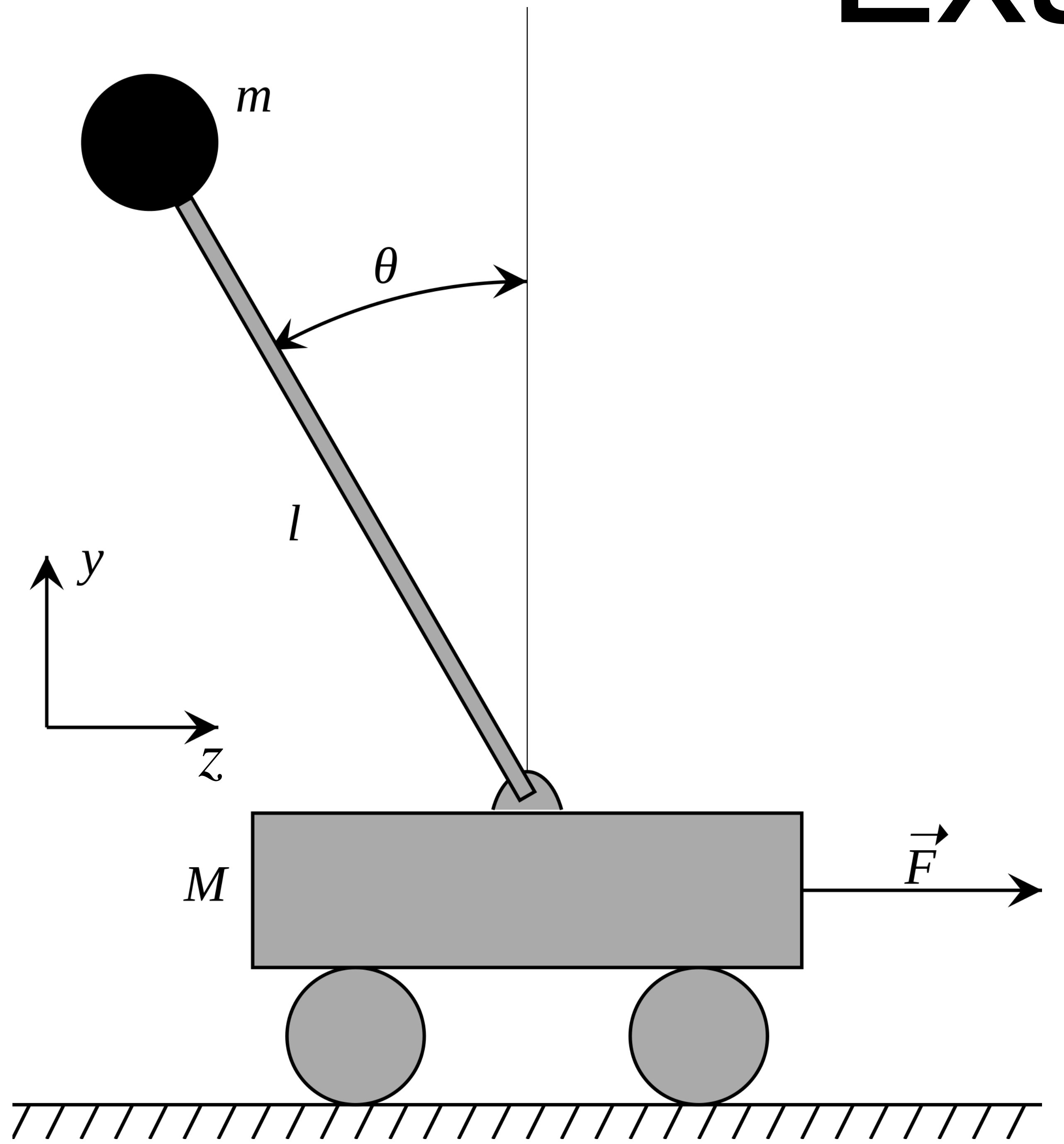

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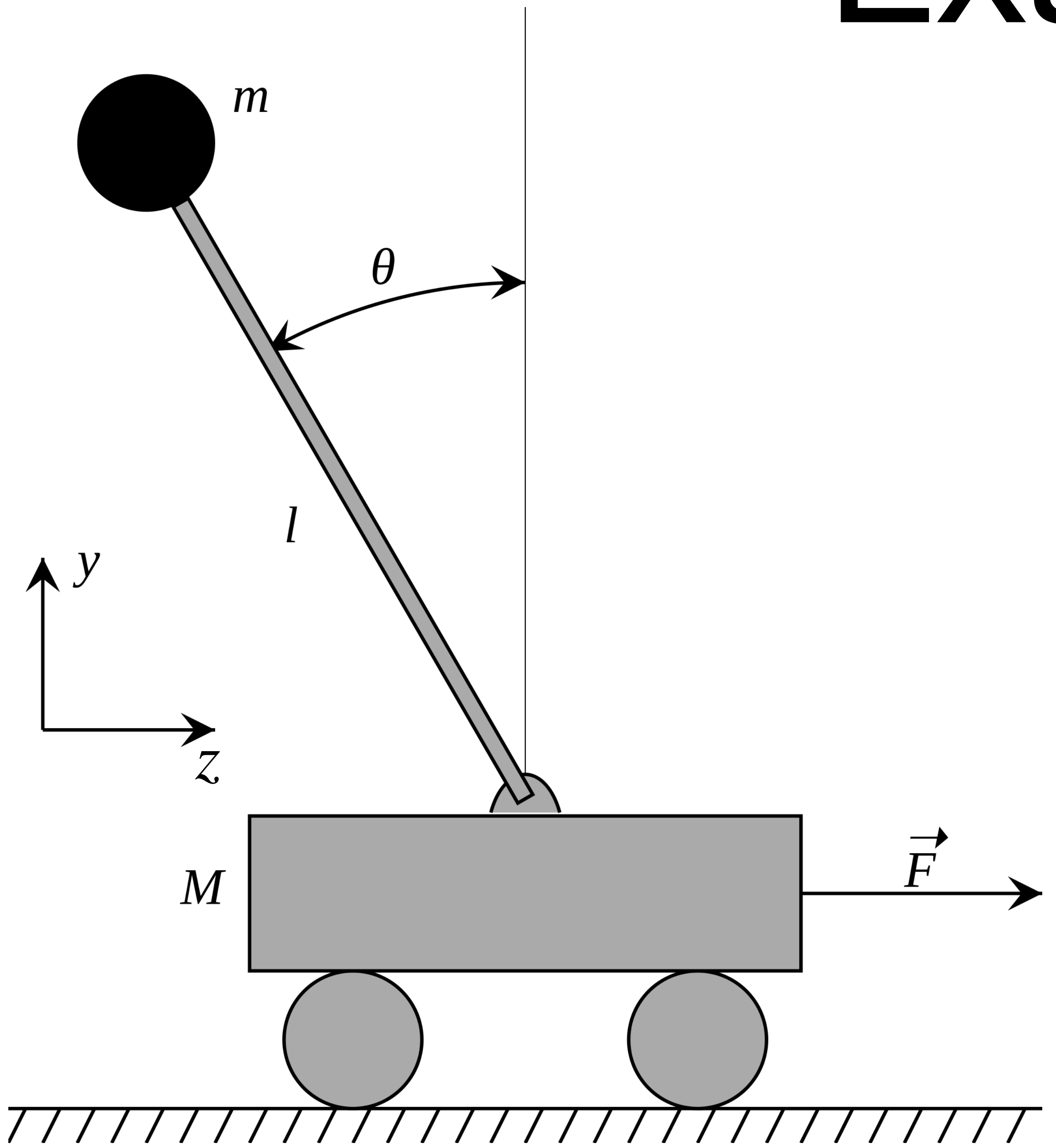

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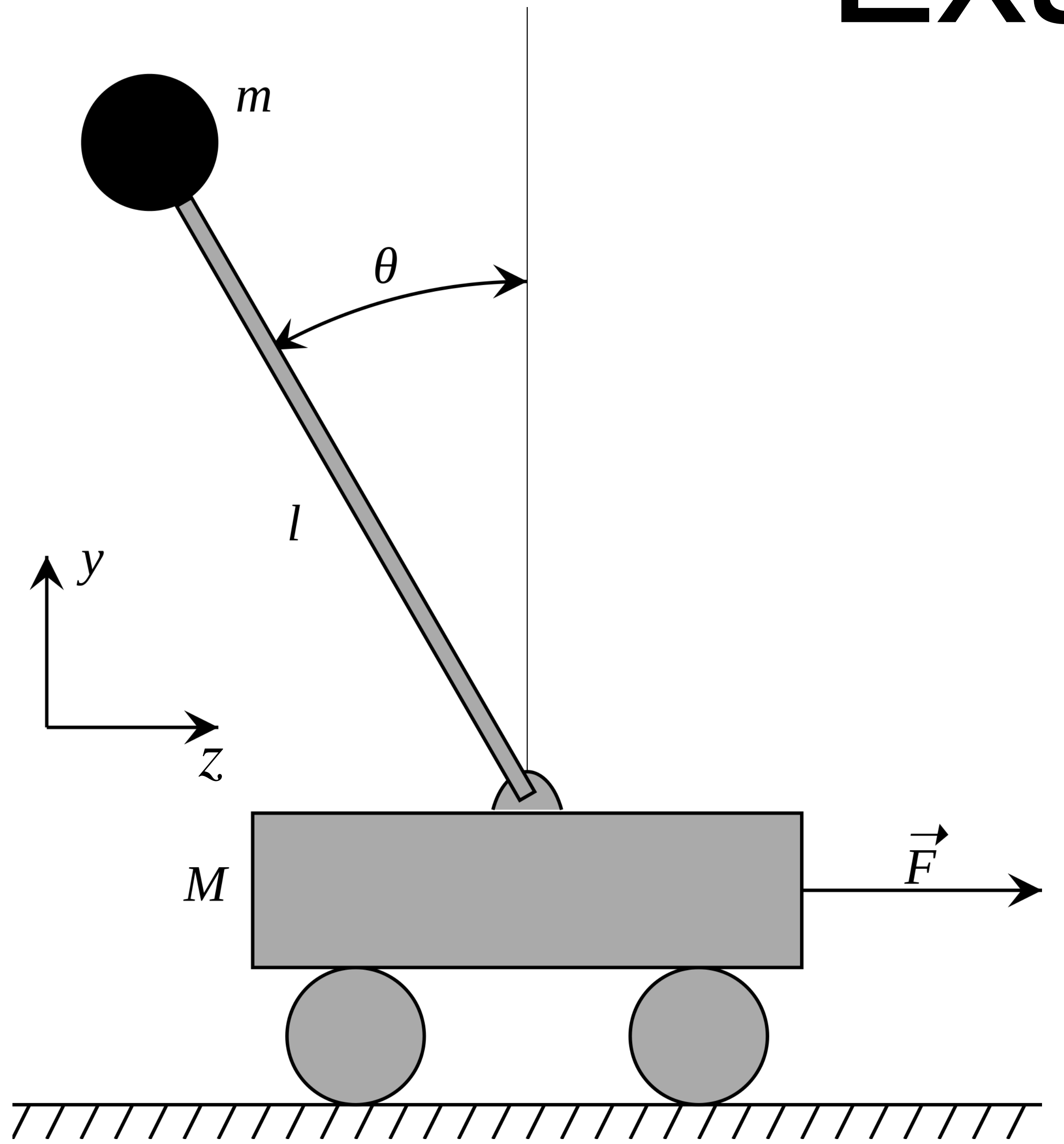

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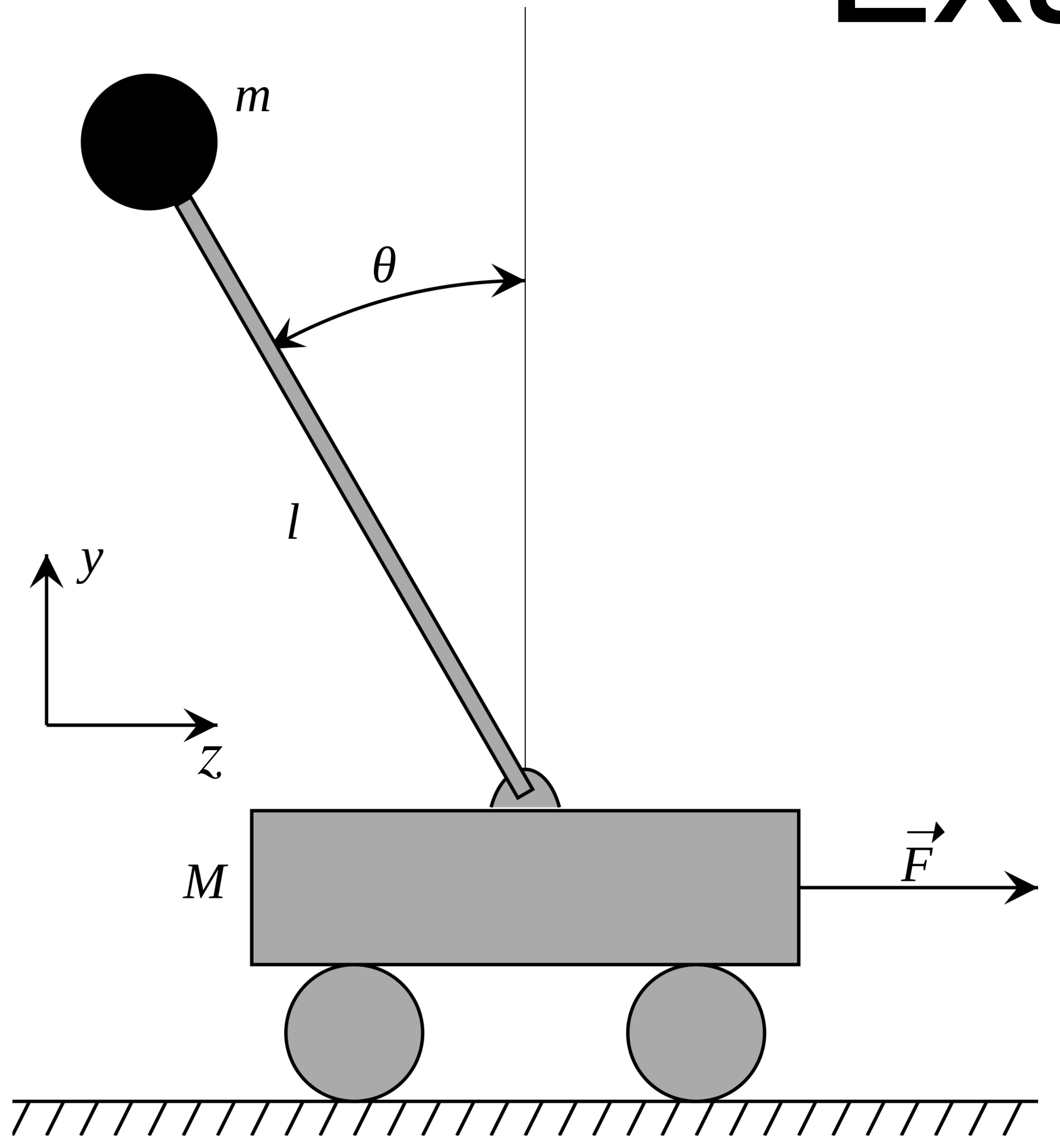

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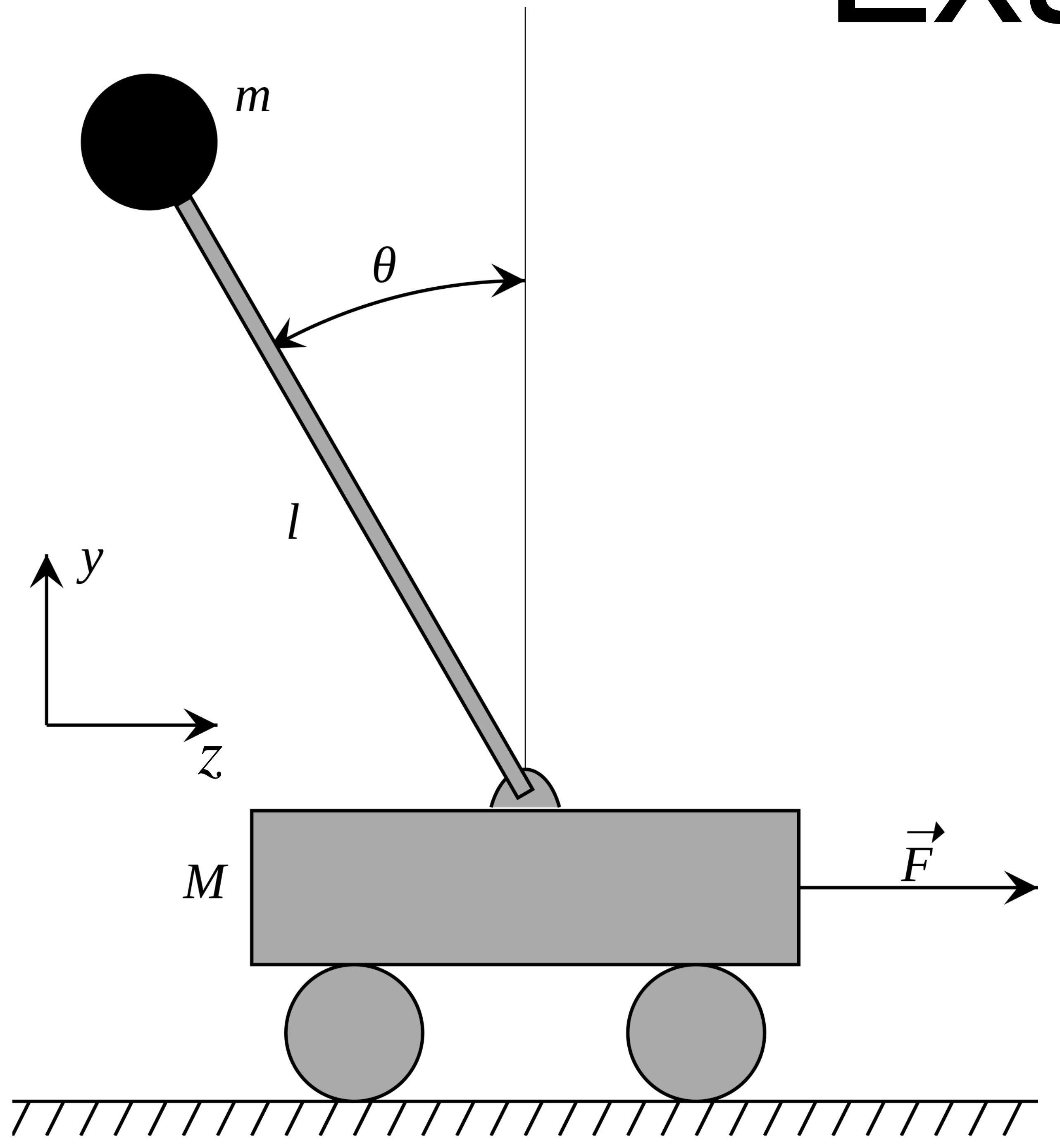

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$$\min_{\pi_0,\ldots,\pi_{H-1}: X \to U} \mathbb{E}\left[\sum_{h=0}^{H-1} c(x_h, u_h)\right] \quad \text{s.t. } x_{h+1} = f(x_h, u_h),\, x_0 \sim \mu_0$$

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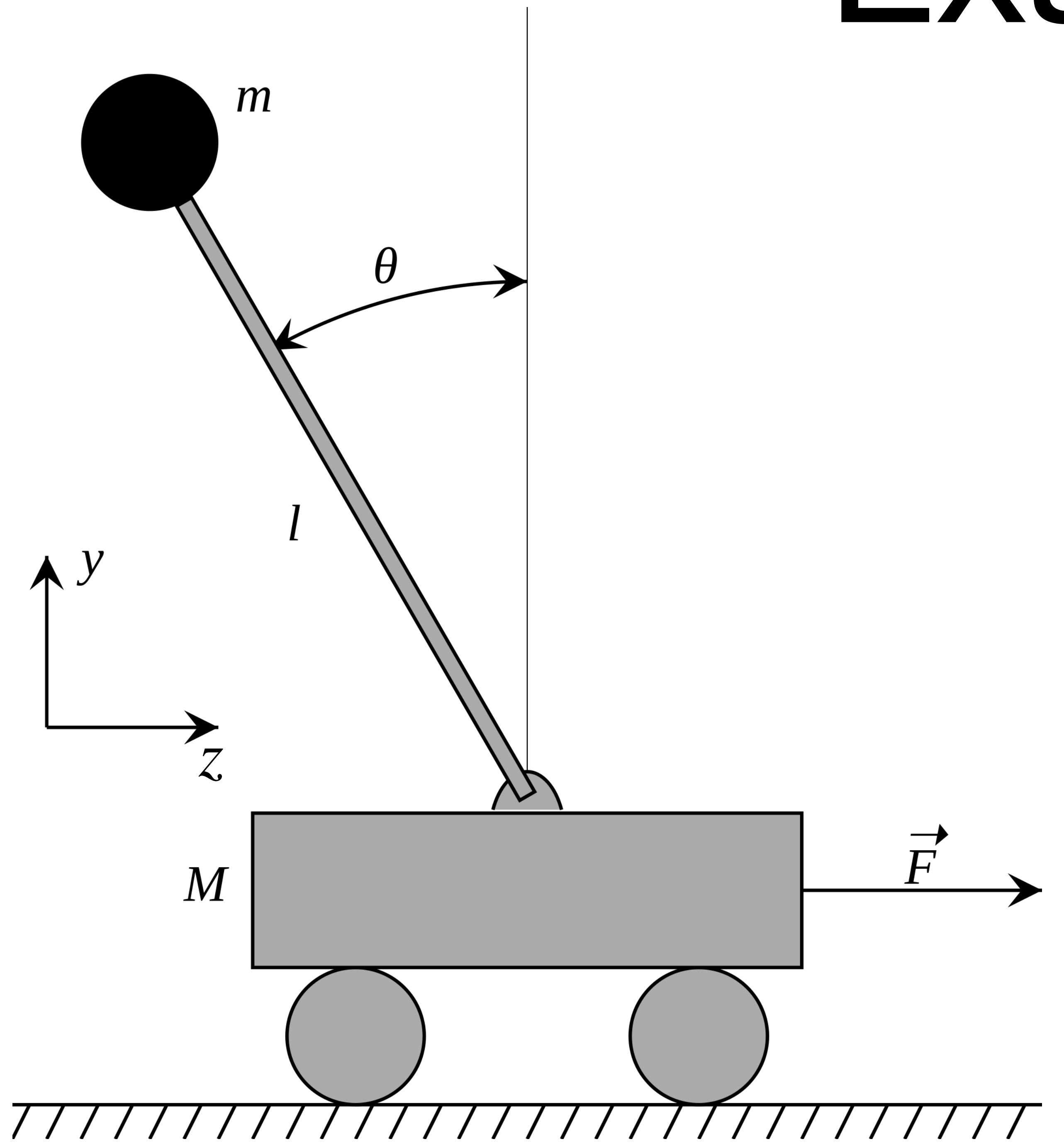

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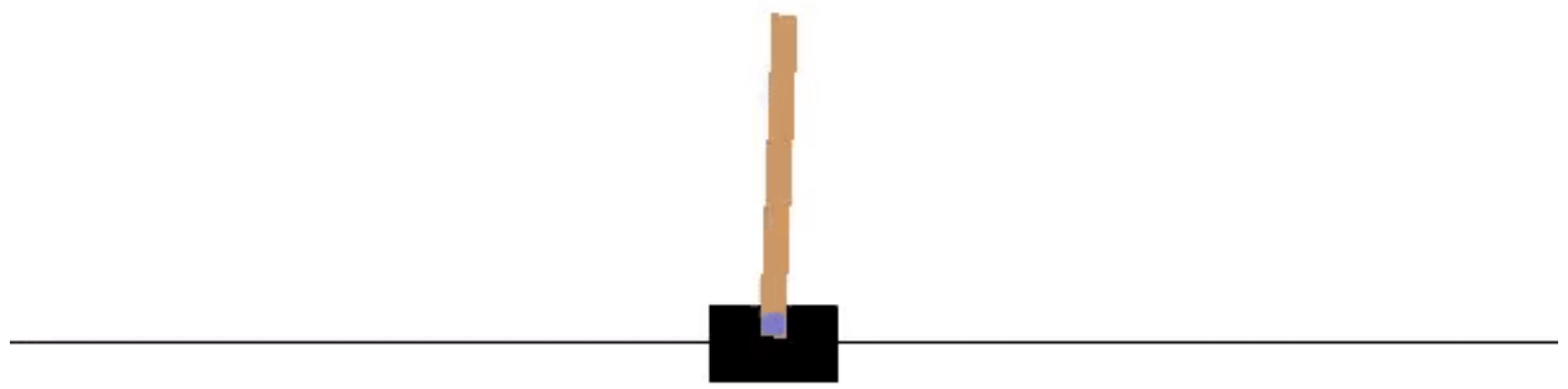

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- Note c_H separated out because by convention there is no u_H

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So why not rely on this more formally by assuming smoothness/structure on the dynamics f and cost c?

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- Note lack of subscripts on c (except at H) and f: time-homogeneous

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That said, it is indeed far too simple for many more complex (nonlinear) systems, though next lecture we will see how to extend it to some nonlinear systems to get surprisingly good solutions

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If time steps are separated by δ (small), then we can approximate acceleration (derivative of velocity) by finite difference of velocities v_h:

$$\text{acceleration}_h = \frac{v_h - v_{h-1}}{\delta} = \frac{u_h}{m}$$

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If time steps are separated by δ (small), then we can approximate acceleration (derivative of velocity) by finite difference of velocities v_h:

$$\text{acceleration}_h = \frac{v_h - v_{h-1}}{\delta} = \frac{u_h}{m}$$

Same trick to approximate velocity (derivative of position) via positions p_h:

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Example: 1-d Vehicle

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So if state $x_h = (p_h, v_h)$, we basically get linear dynamics!

LQR Value and Q functions

LQR Value and Q functions

Given a policy $\pi = (\pi_0, \ldots, \pi_{h-1})$, define the value function $V_h^\pi : \mathbb{R}^d \to \mathbb{R}$ as:

$$V_h^\pi(x) = \mathbb{E}\left[x_H^\top Q x_H + \sum_{i=h}^{H-1} (x_i^\top Q x_i + u_i^\top R u_i) \,\middle|\, u_i = \pi_i(x_i)\ \forall i \geq h,\ x_h = x\right]$$

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and the Q function $Q_h^\pi : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ as:

$$Q_h^\pi(x, u) = \mathbb{E}\left[x_H^\top Q x_H + \sum_{i=h}^{H-1} (x_i^\top Q x_i + u_i^\top R u_i) \;\middle|\; u_h = u,\ u_i = \pi_i(x_i)\ \forall i > h,\ x_h = x\right]$$

Today

- ✓ Feedback from last lecture
- ✓ Recap
- ✓ General optimal control problem
- ✓ The linear quadratic regulator (LQR) problem
- Optimal control solution to LQR

LQR Optimal Control

LQR Optimal Control

$$V_h^\star(x) = \min_\pi V_h^\pi(x) = \min_{\pi_h, \pi_{h+1}, \ldots, \pi_{H-1}} \mathbb{E}\Big[x_H^\top Q x_H + \sum_{i=h}^{H-1} (x_i^\top Q x_i + u_i^\top R u_i) \;\Big|\; u_i = \pi_i(x_i)\ \forall i \geq h,\ x_h = x\Big]$$

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Theorem:

1. $V_h^\star$ is a quadratic function, i.e., $V_h^\star(x) = x^\top P_h x + p_h$ for some $P_h \in \mathbb{R}^{d\times d}$ and $p_h \in \mathbb{R}^d$
2. The optimal policy $\pi_h^\star$ is linear, i.e., $\pi_h^\star(x) = -K_h x$ for some $K_h \in \mathbb{R}^{k\times d}$
3. P_h, p_h, and K_h can be computed exactly

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We will cover the steps of the proof the theorem and derive the optimal policy along the way via dynamic programming

Key Steps in the Proof

Dynamic programming (finite-horizon), stepping backwards in time from H to 0

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   a) Show that $Q_h^\star(x,u)$ is quadratic (in both x and u)
   b) Derive the optimal policy $\pi_h^\star(x) = \arg\min_u Q_h^\star(x,u)$, and show that it's linear
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   c) Show $V_h^\star(x)$ is quadratic

3. **Conclusion:** $V_h^\star(x)$ is quadratic and $\pi_h^\star(x)$ is linear and we'll have their formulas

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Recall the value function at a given h is:

$$V_h^\pi(x) = \mathbb{E}\left[x_H^\top Q x_H + \sum_{i=h}^{H-1} (x_i^\top Q x_i + u_i^\top R u_i) \;\middle|\; u_i = \pi_i(x_i)\ \forall i \geq h,\ x_h = x\right]$$

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(P_h and p_h didn't do much here, but we're going to define them recursively in the next step)

Induction Step

Assume $V^\star_{h+1}(x) = x^\top P_{h+1} x + p_{h+1}$, for all x, where $P_{h+1} \in \mathbb{R}^{d \times d}$ and $p_{h+1} \in \mathbb{R}^d$

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Induction Step (continued)

$$\begin{aligned} Q_h^\star(x,u) &= c(x,u) + \mathbb{E}_{x' \sim f(x,u,w_{h+1})}\left[V_{h+1}^\star(x')\right] \\ &= x^\top \left(Q + A^\top P_{h+1} A\right) x + u^\top \left(R + B^\top P_{h+1} B\right) u + 2x^\top A^\top P_{h+1} B u + \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1} \end{aligned}$$

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Set $\nabla_u Q_h^\star(x,u) = 0$ and solve for u:

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$$P_h = Q + A^\top P_{h+1} A - A^\top P_{h+1} B (R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A$$
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$$\pi_h^\star(x) = \underbrace{-(R + B^\top P_{h+1}B)^{-1}B^\top P_{h+1}A}_{:=K_h}\,x$$

$$V_h^\star(x) = Q_h^\star(x, \pi_h^\star(x))$$

$$= x^\top\left(Q + A^\top P_{h+1}A\right)x + x^\top K_h^\top\left(R + B^\top P_{h+1}B\right)K_h x - 2x^\top A^\top P_{h+1}BK_h x + \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

Collecting the quadratic and constant terms together, $V_h^\star(x) = x^\top P_h x + p_h$, where:

$$P_h = Q + A^\top P_{h+1}A - A^\top P_{h+1}B(R + B^\top P_{h+1}B)^{-1}B^\top P_{h+1}A$$ ← **Ricatti Equation**

$$p_h = \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

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Optimal policy has nothing to do with initial distribution μ_0 or the noise σ^2!

Today

- Feedback from last lecture
- Recap
- General optimal control problem
- The linear quadratic regulator (LQR) problem
- Optimal control solution to LQR

Summary:

- Optimal control: Find optimal policy in MDP with continuous state/action spaces
- Linear quadratic regulator (LQR) is canonical problem in optimal control
  - Linear dynamics, Gaussian errors, quadratic costs
  - Optimal value and policy follow from dynamic programming

Attendance:
bit.ly/3RcTC9T

Feedback:
bit.ly/3RHtlxy