Dynamic Programming & Infinite Horizons

Lucas Janson

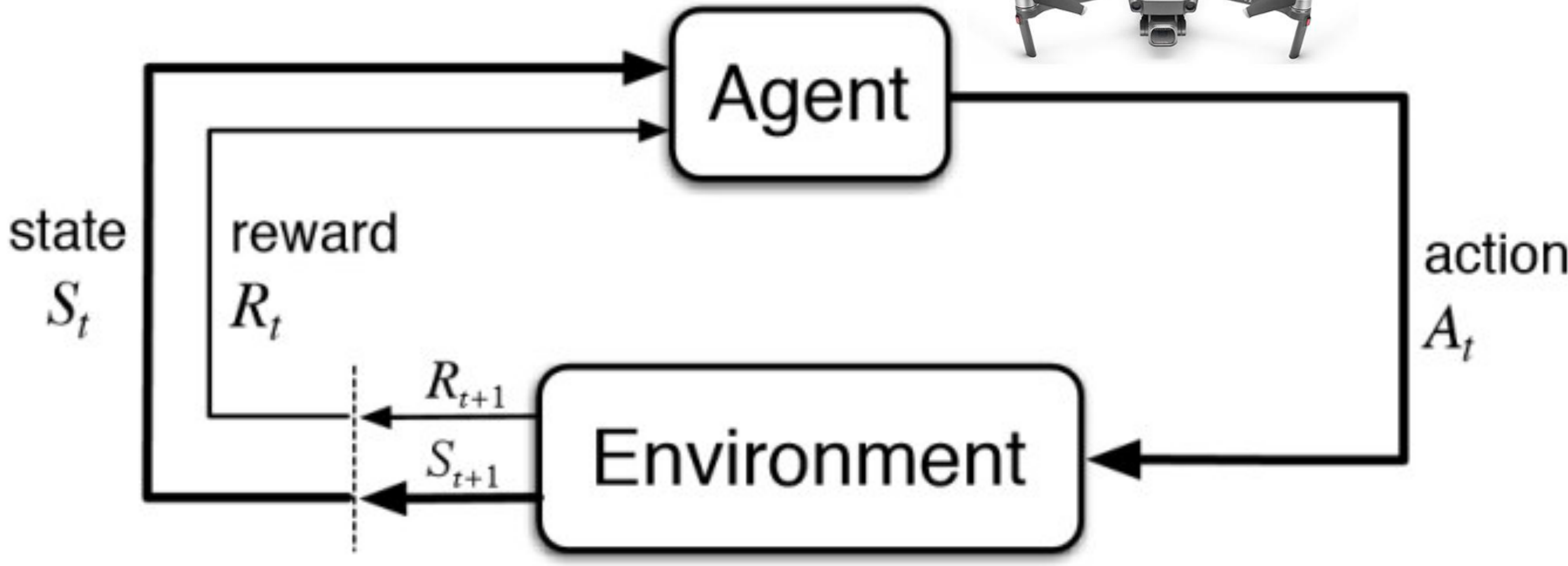
CS/Stat 184(0): Introduction to Reinforcement Learning

Fall 2024

Today

- Recap
- Optimality
- The Bellman Equations & Dynamic Programming
- Infinite Horizons

Finite Horizon Markov Decision Processes (MDPs):



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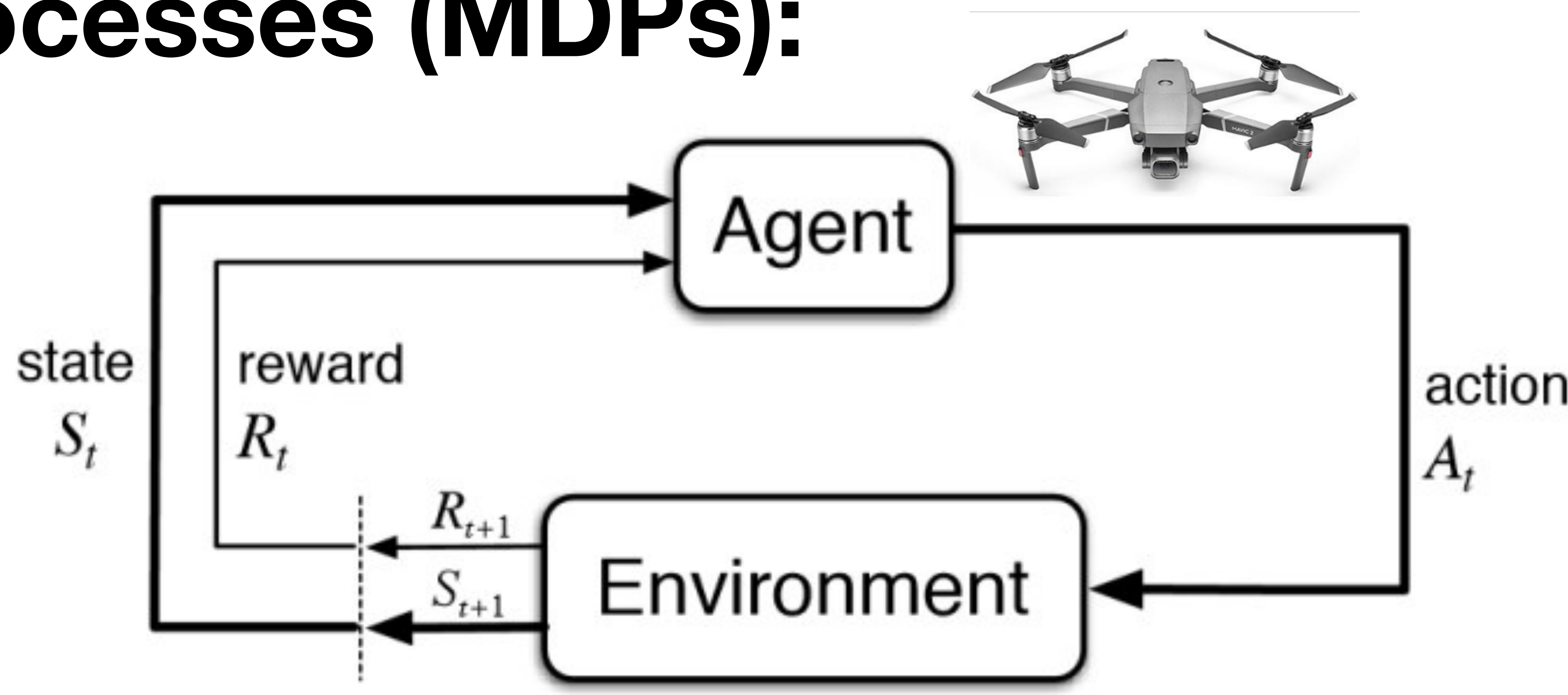

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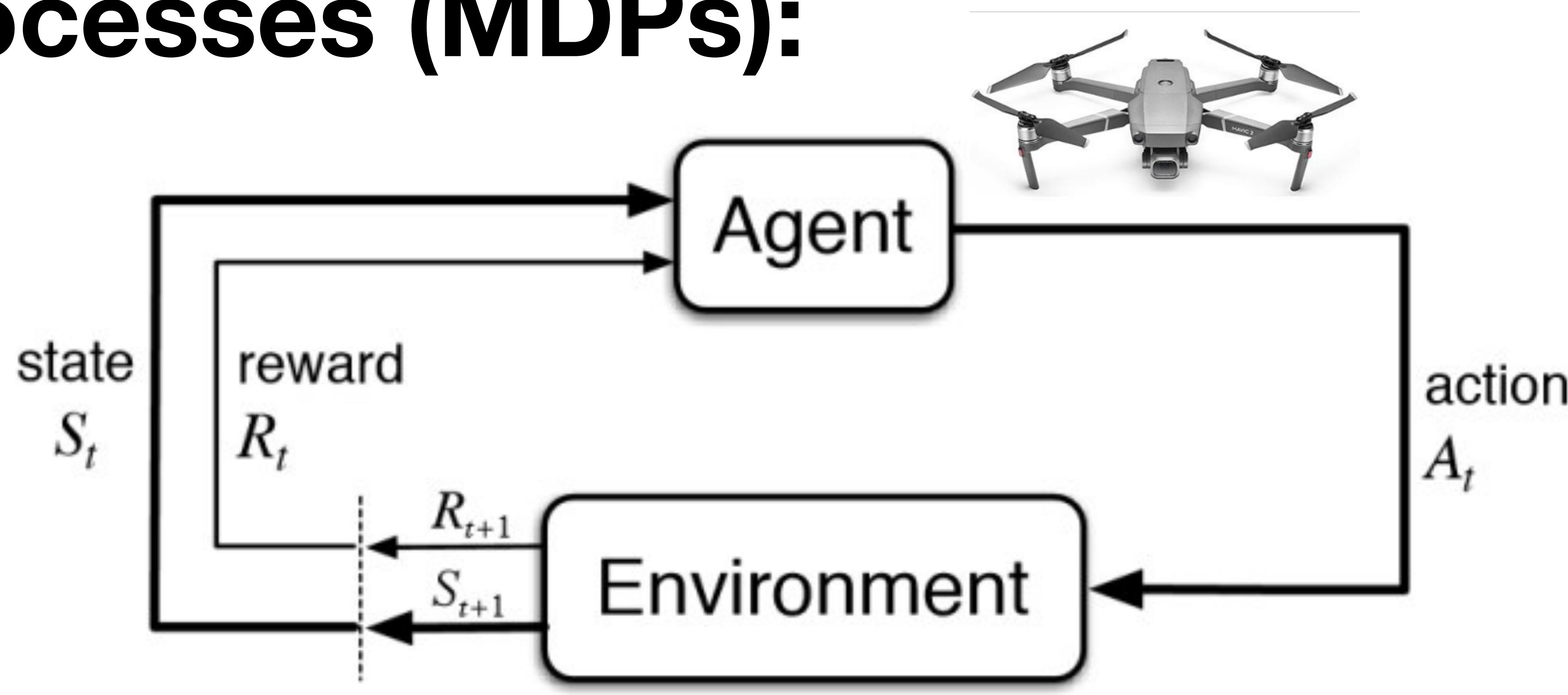

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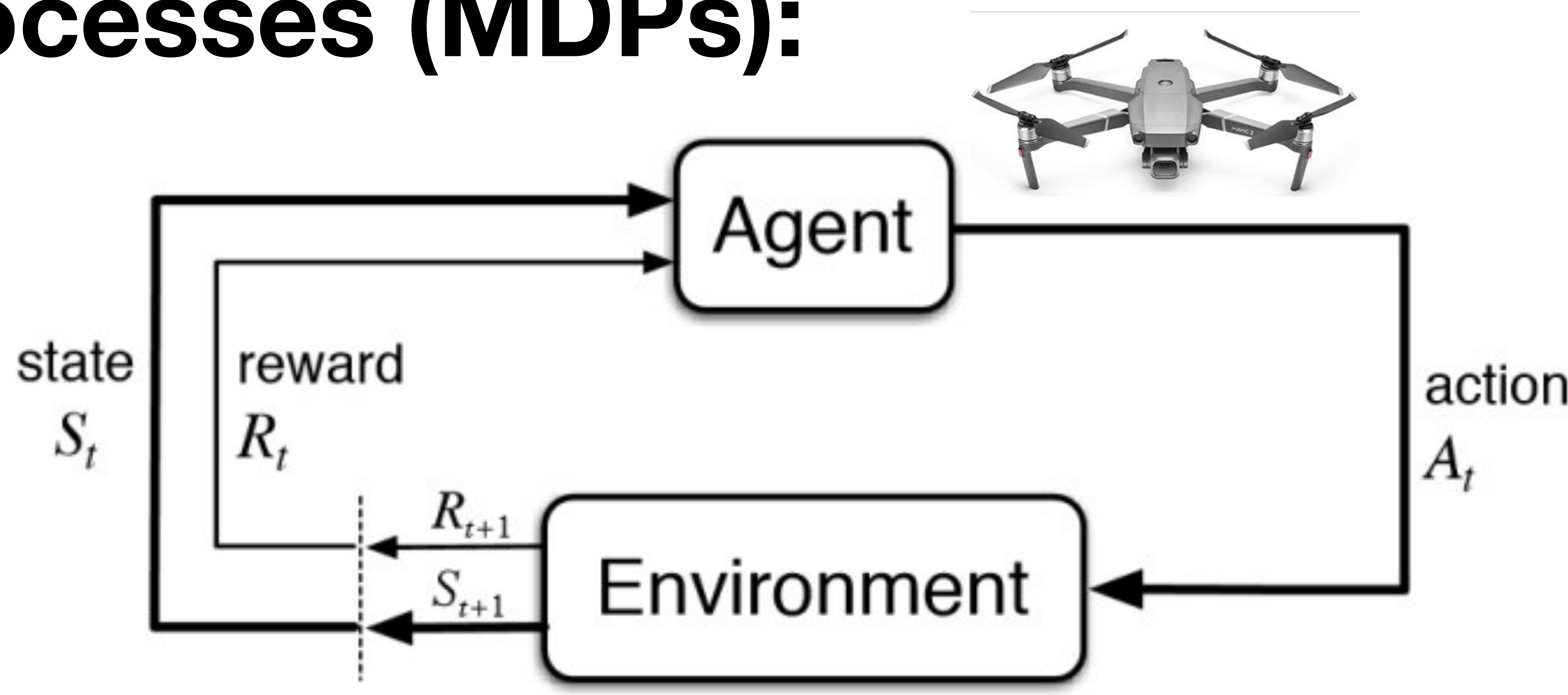

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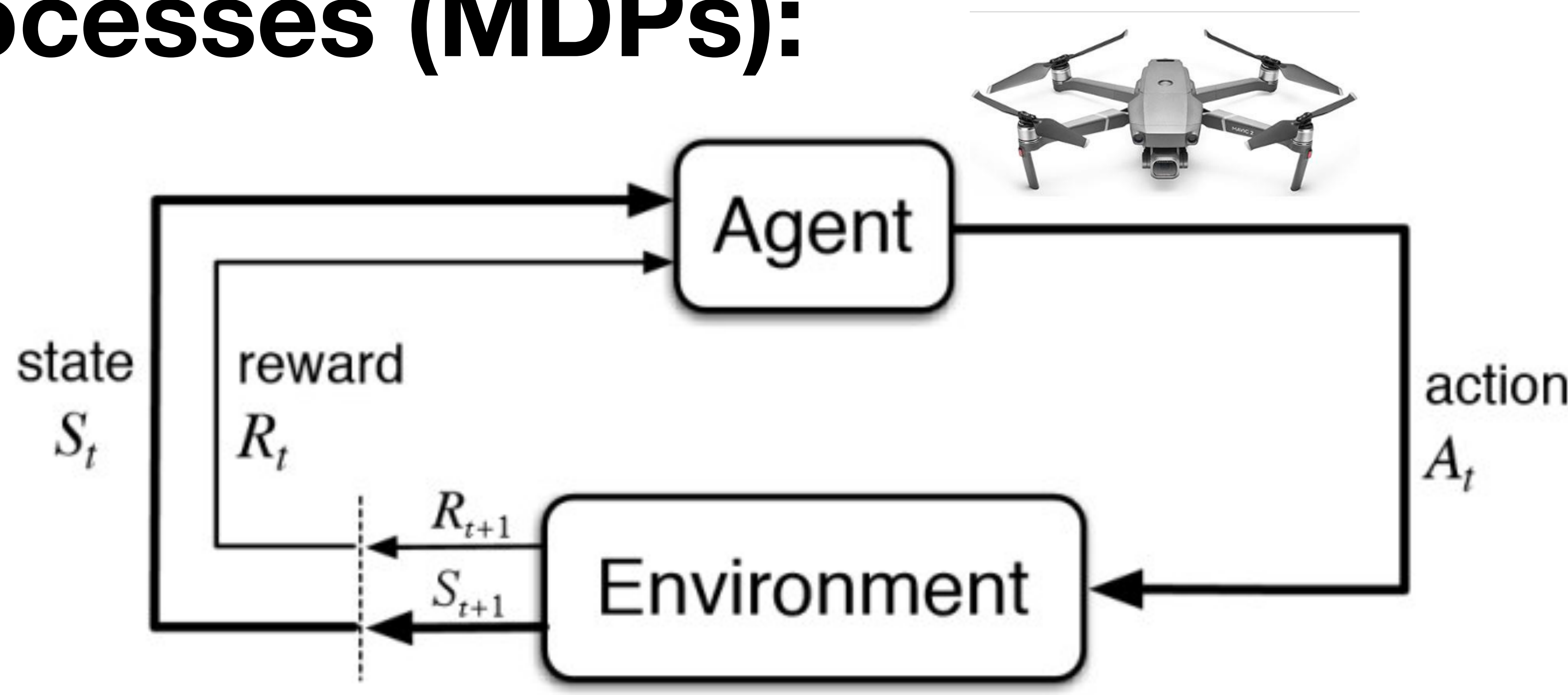

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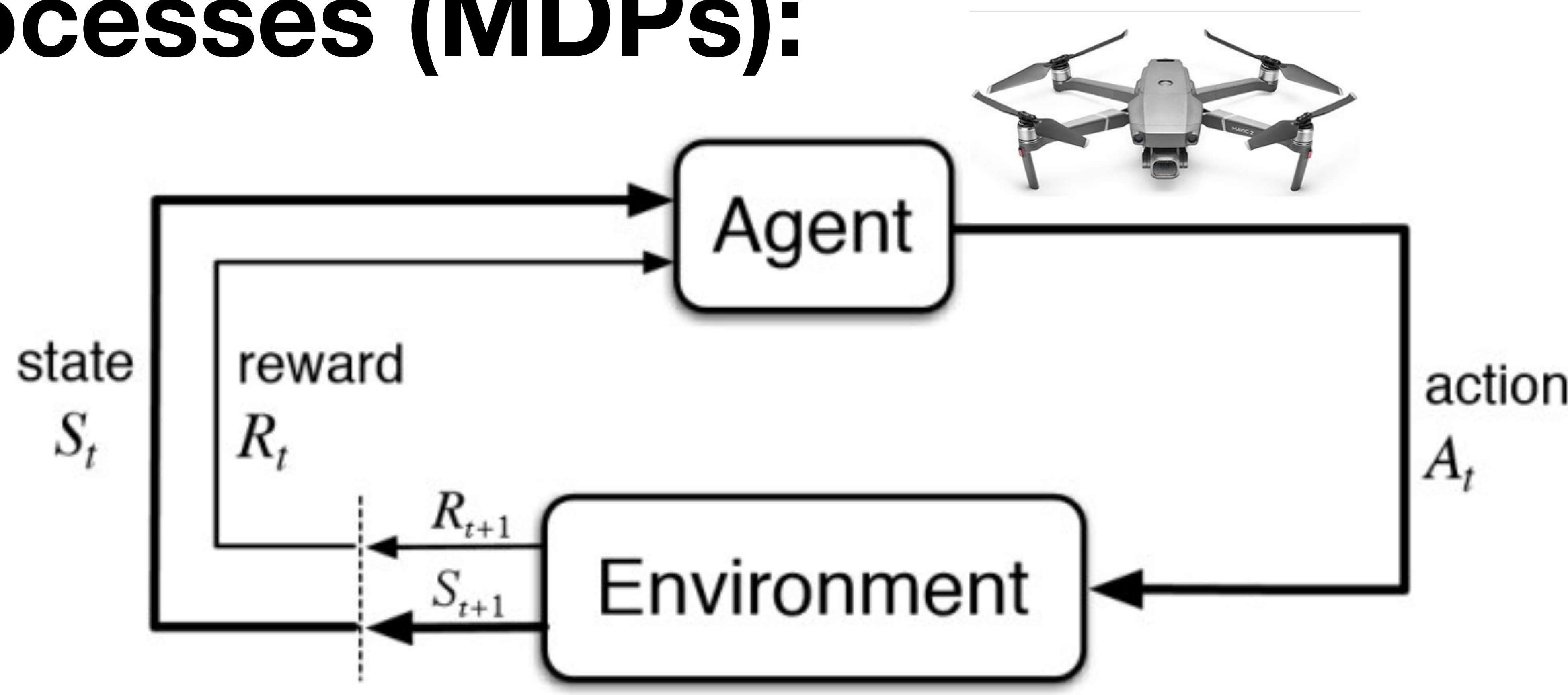

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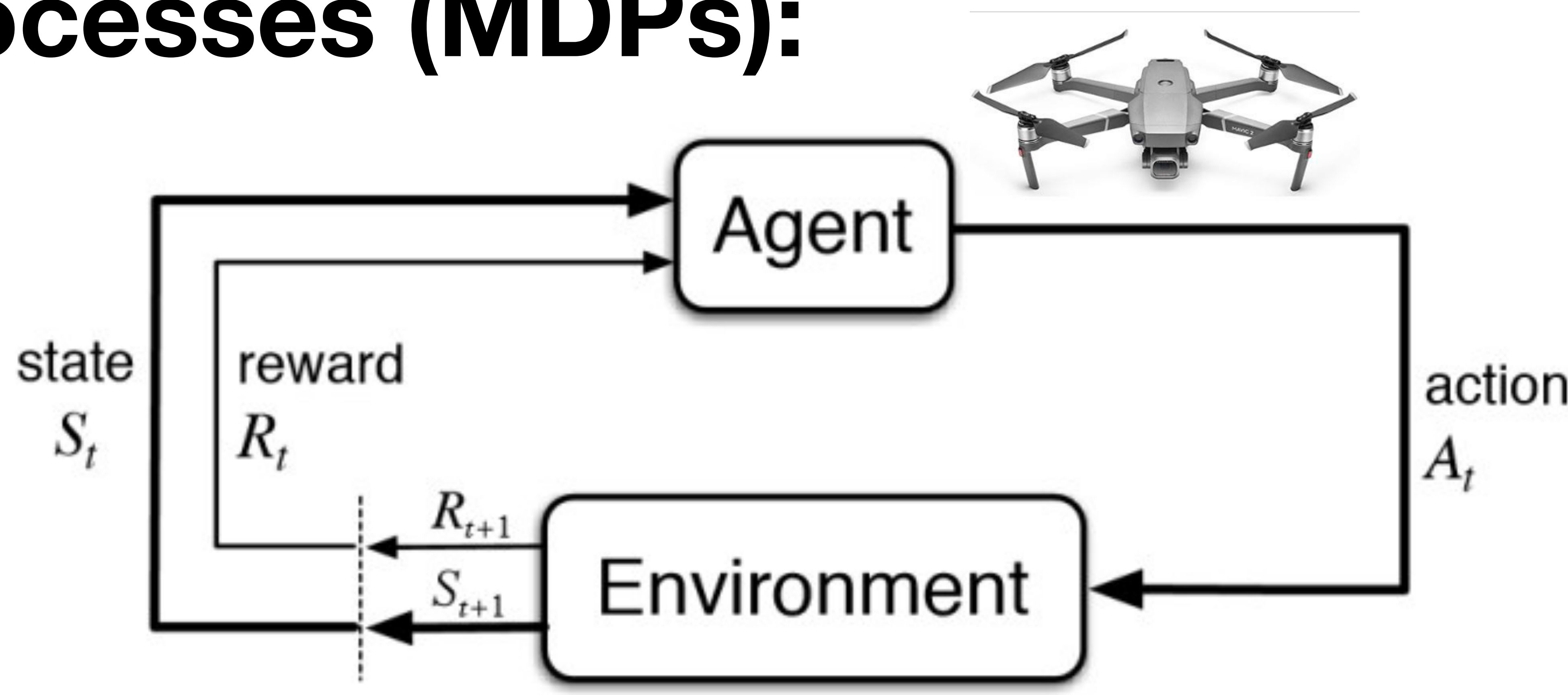

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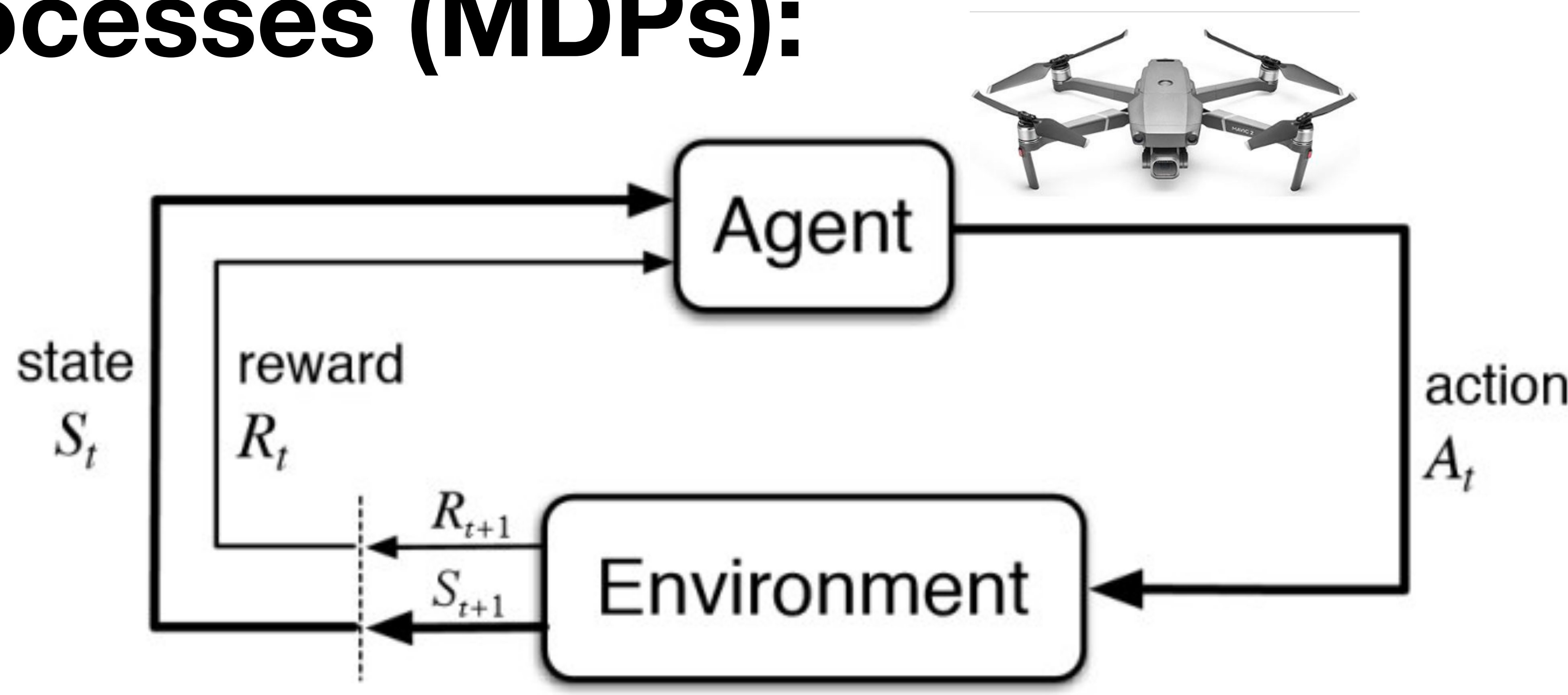

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  - A time horizon $H \in \mathbb{N}$

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- For deterministic policy π, Bellman consistency:
  - $V_h^\pi(s) = r(s, \pi_h(s)) + \mathbb{E}_{s' \sim P(\cdot|s, \pi_h(s))}\left[V_{h+1}^\pi(s')\right]$
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  - DP:
    - Initialize: $V_H^\pi(s) = 0, \ \forall s \in S$
    - For $h = H-1, \ldots 0$, set:
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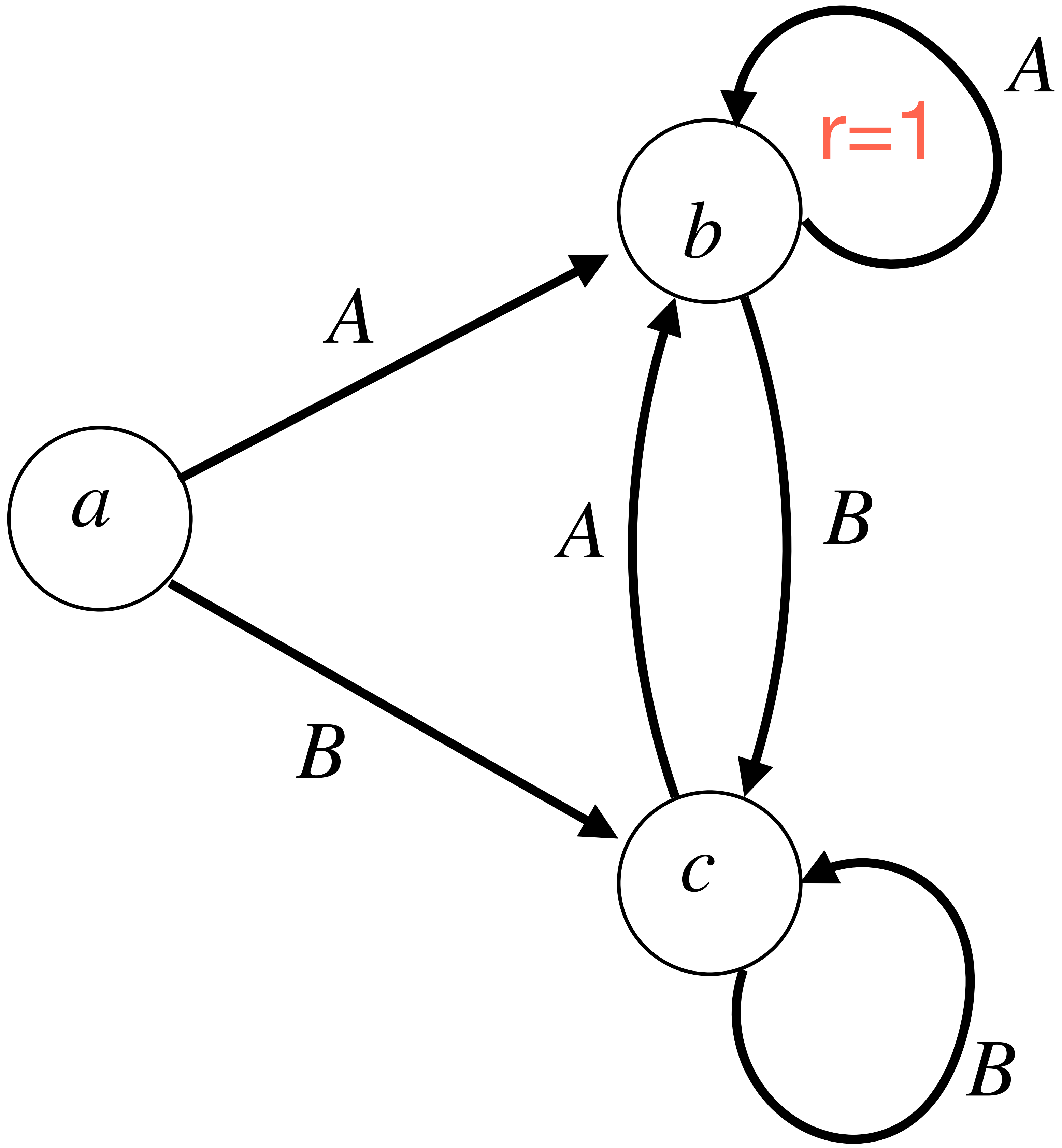

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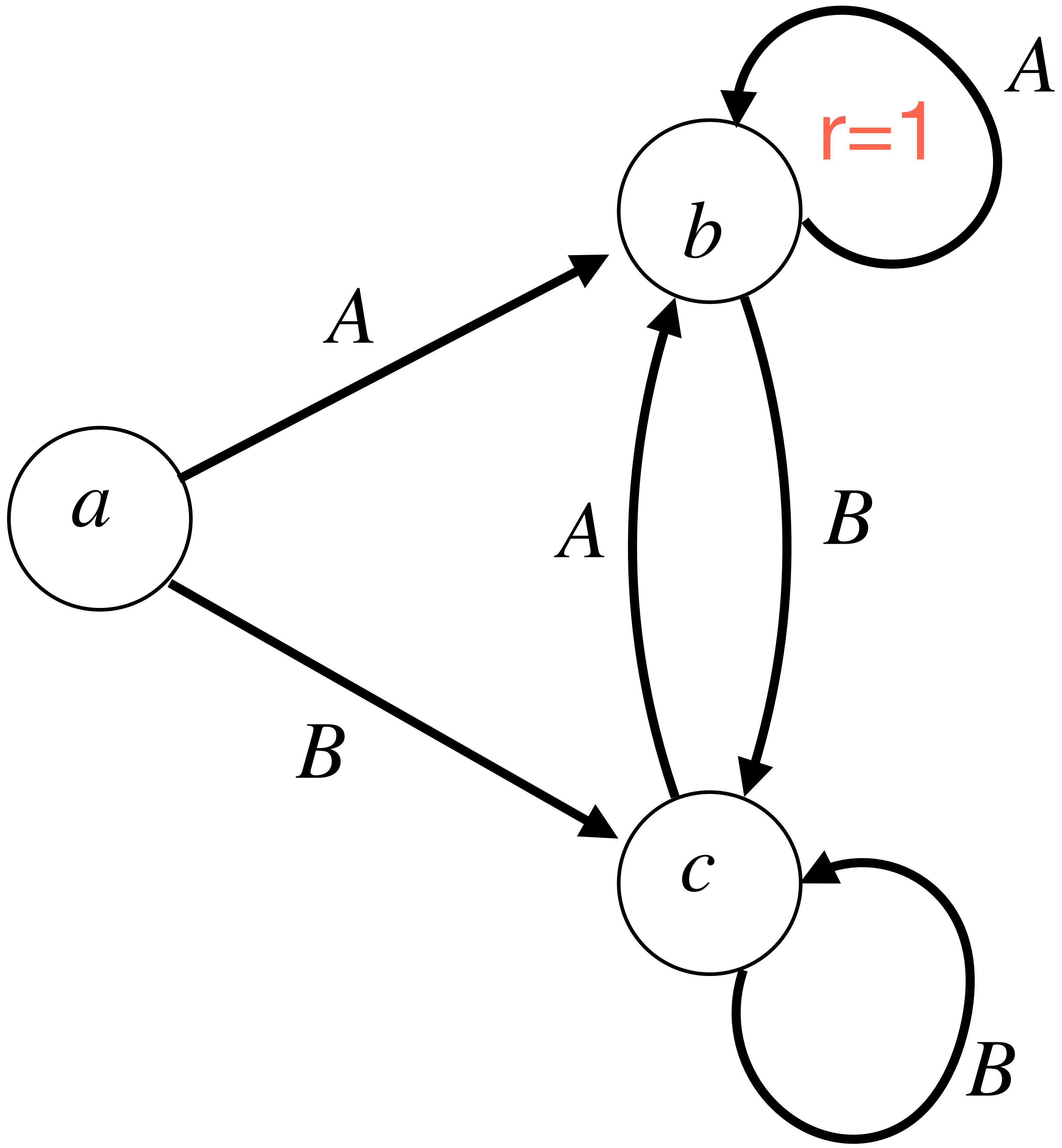

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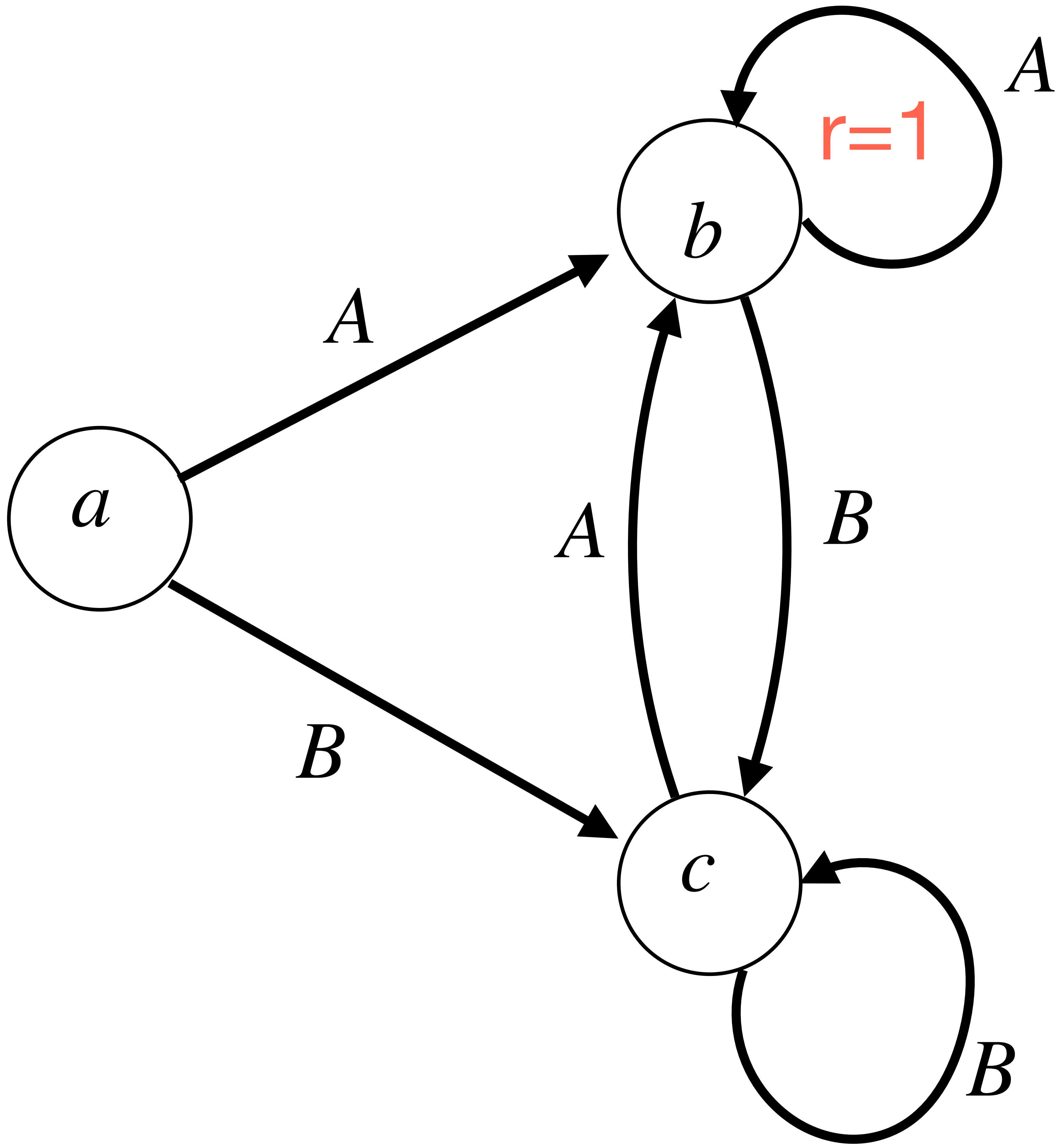

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- What is optimal value function, $V^{\pi^\star} = V^\star$?
  $V_2^\star(a) = 0, \ V_2^\star(b) = 1, \ V_2^\star(c) = 0$

  $V_1^\star(a) = 1, \ V_1^\star(b) = 2, \ V_1^\star(c) = 1$

  $V_0^\star(a) = 2, \ V_0^\star(b) = 3, \ V_0^\star(c) = 2$

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$$|A|^{|S|H}$$

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- Can we do better?

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- $\implies$ $\pi^\star$ doesn't depend on the initial state distribution μ.

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- Caveat: some legitimate reward functions are not additive/linear (so, naively, not an MDP). (But, RL is general: think about redefining the state so you can do these.)

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- Objective: find policy π that maximizes our expected, discounted future reward:

$$\max_{\pi} \mathbb{E} \left[r(s_0, a_0) + \gamma r(s_1, a_1) + \gamma^2 r(s_2, a_2) + \ldots\ldots \middle| s_0 \right]$$

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- The infinite trajectory: $\tau = \{s_0, a_0, r_0, s_1, a_1, r_1, \ldots, \}$

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- What are upper and lower bounds on V^π and Q^π

$$\left[0, \sum_{h=0}^{\infty} \gamma^h = \frac{1}{1-\gamma}\right]$$

Example of Policy Evaluation (e.g. computing V^π and Q^π)

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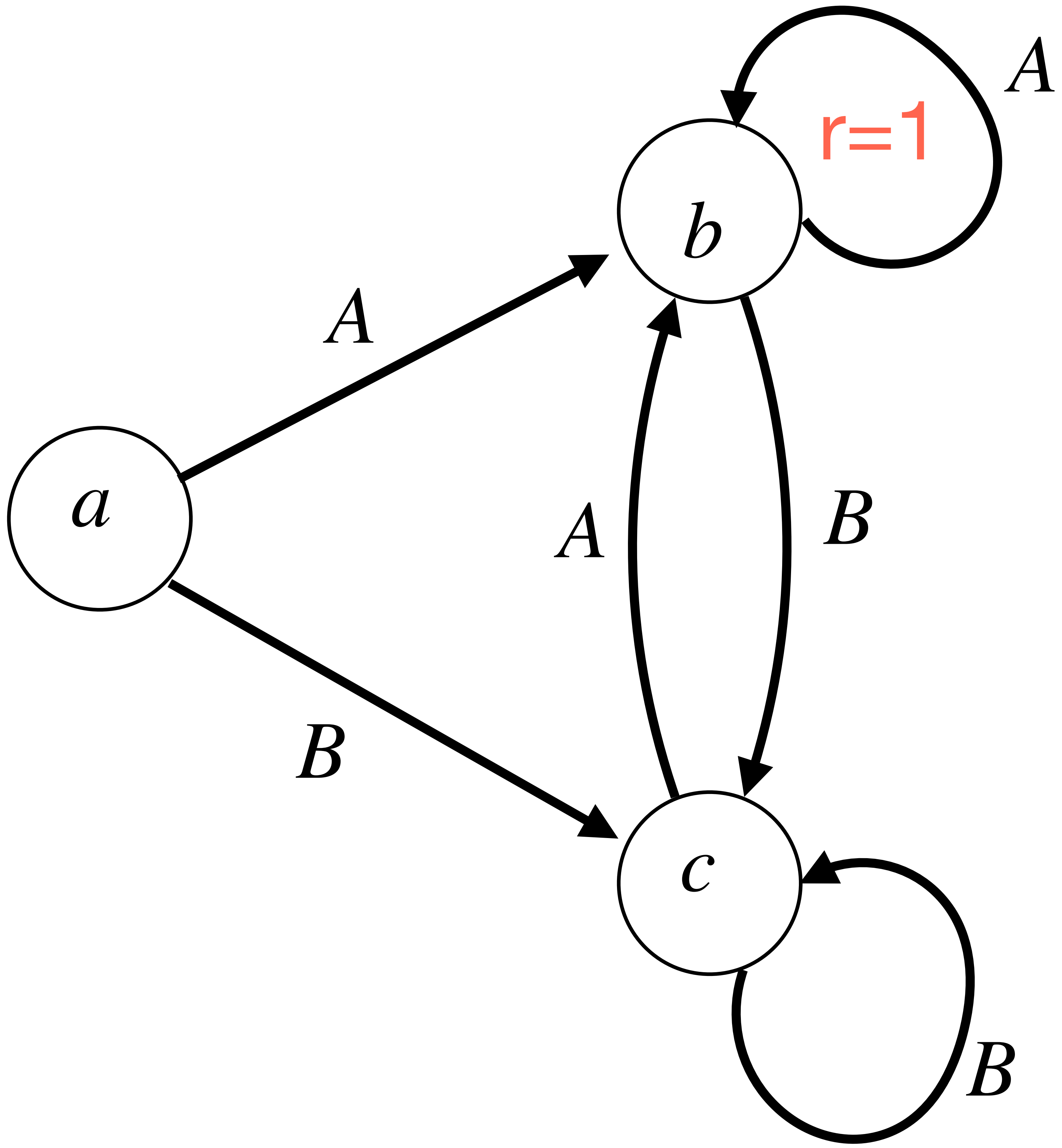

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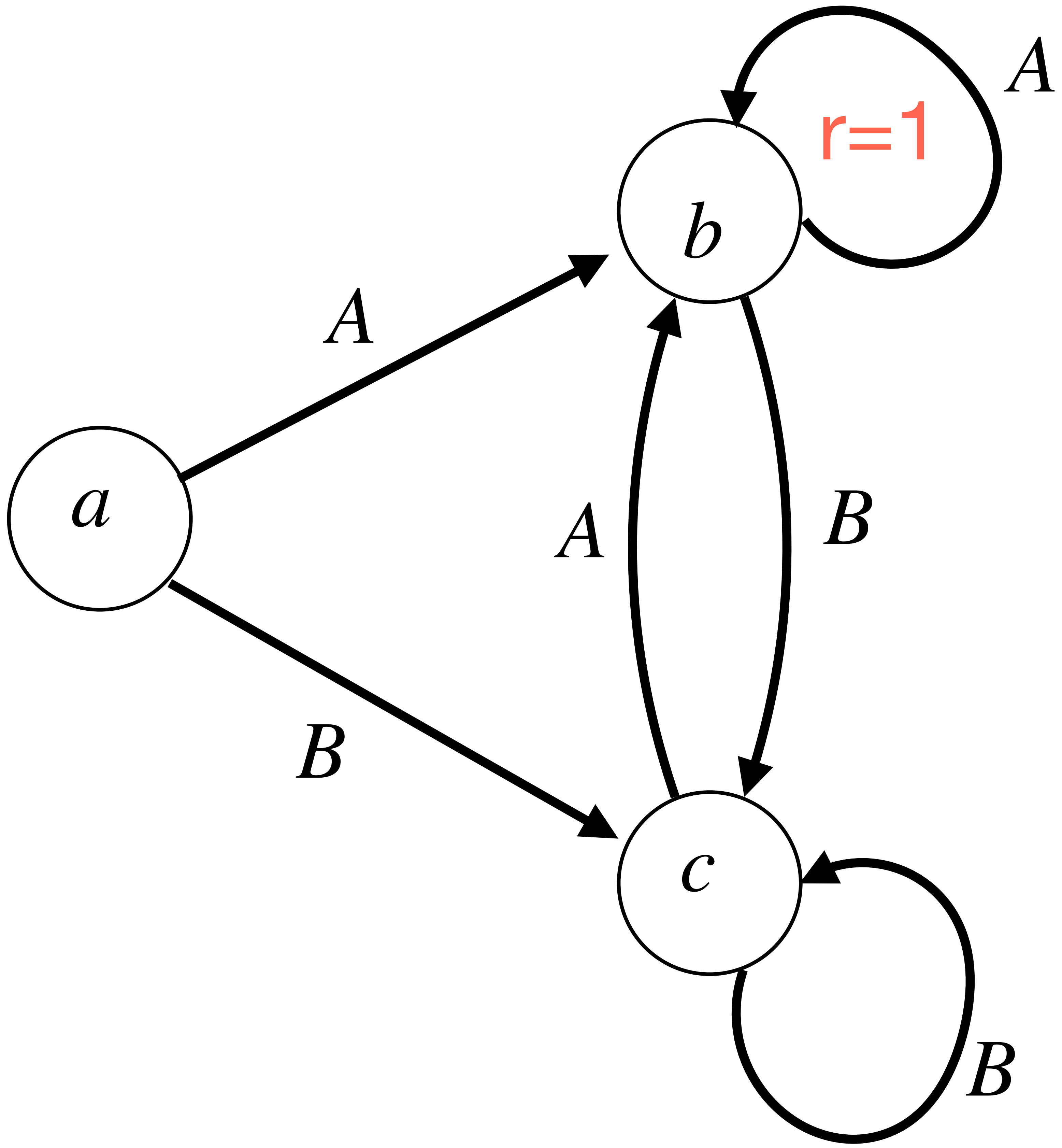

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- Consider the policy
  $\pi(a) = B, \pi(b) = A, \pi(c) = A$

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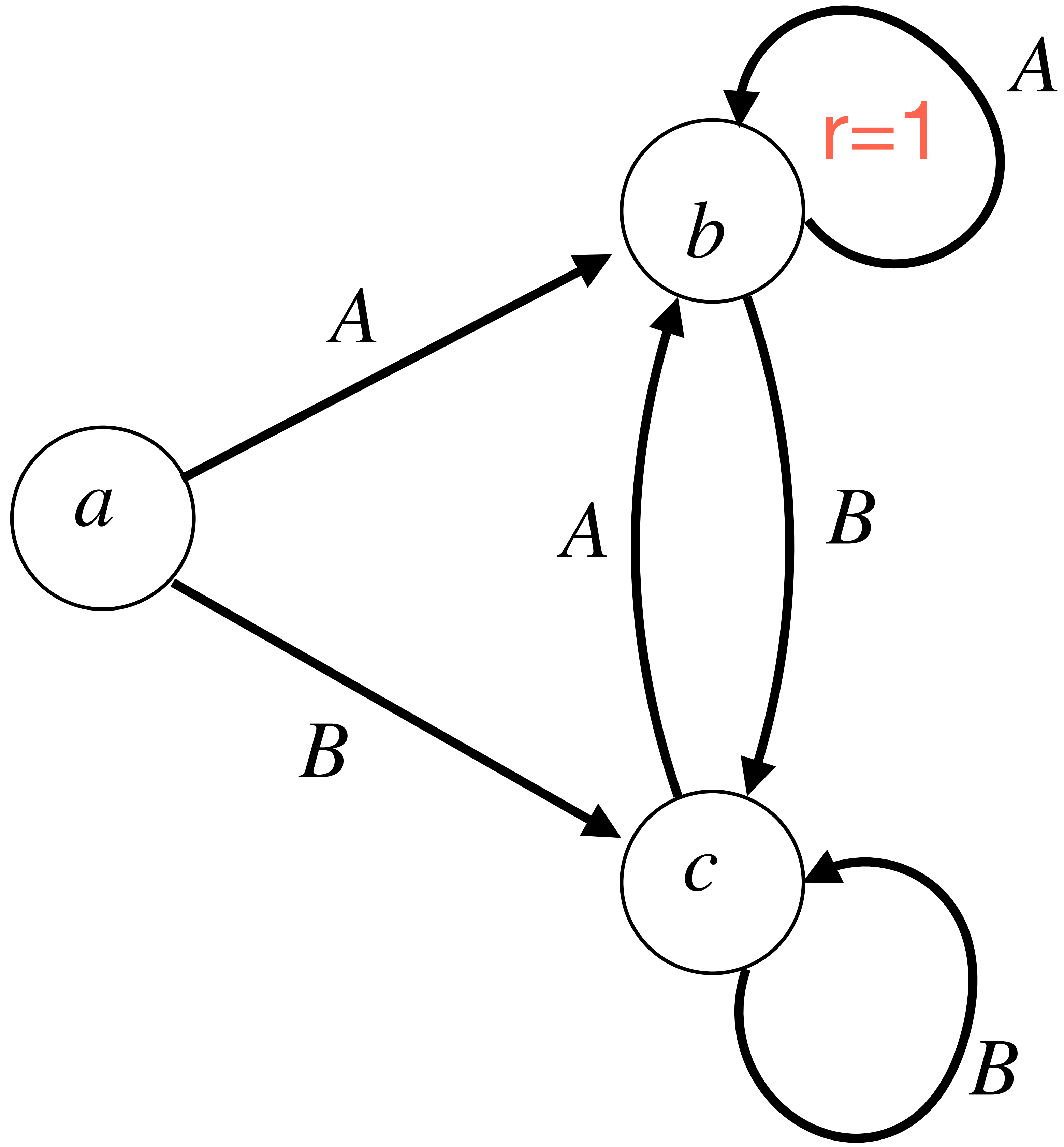

Reward: $r(b, A) = 1$, & 0 everywhere else

- Consider the policy
  $\pi(a) = B, \pi(b) = A, \pi(c) = A$
- What is V^π?

$V^\pi(a) = \sum_{h=2}^{\infty} \gamma^h = \sum_{h'=0}^{\infty} \gamma^{h'+2} = \gamma^2 \sum_{h'=0}^{\infty} \gamma^{h'}$

$\frac{1}{1-\gamma}$

$V^\pi(b) =$

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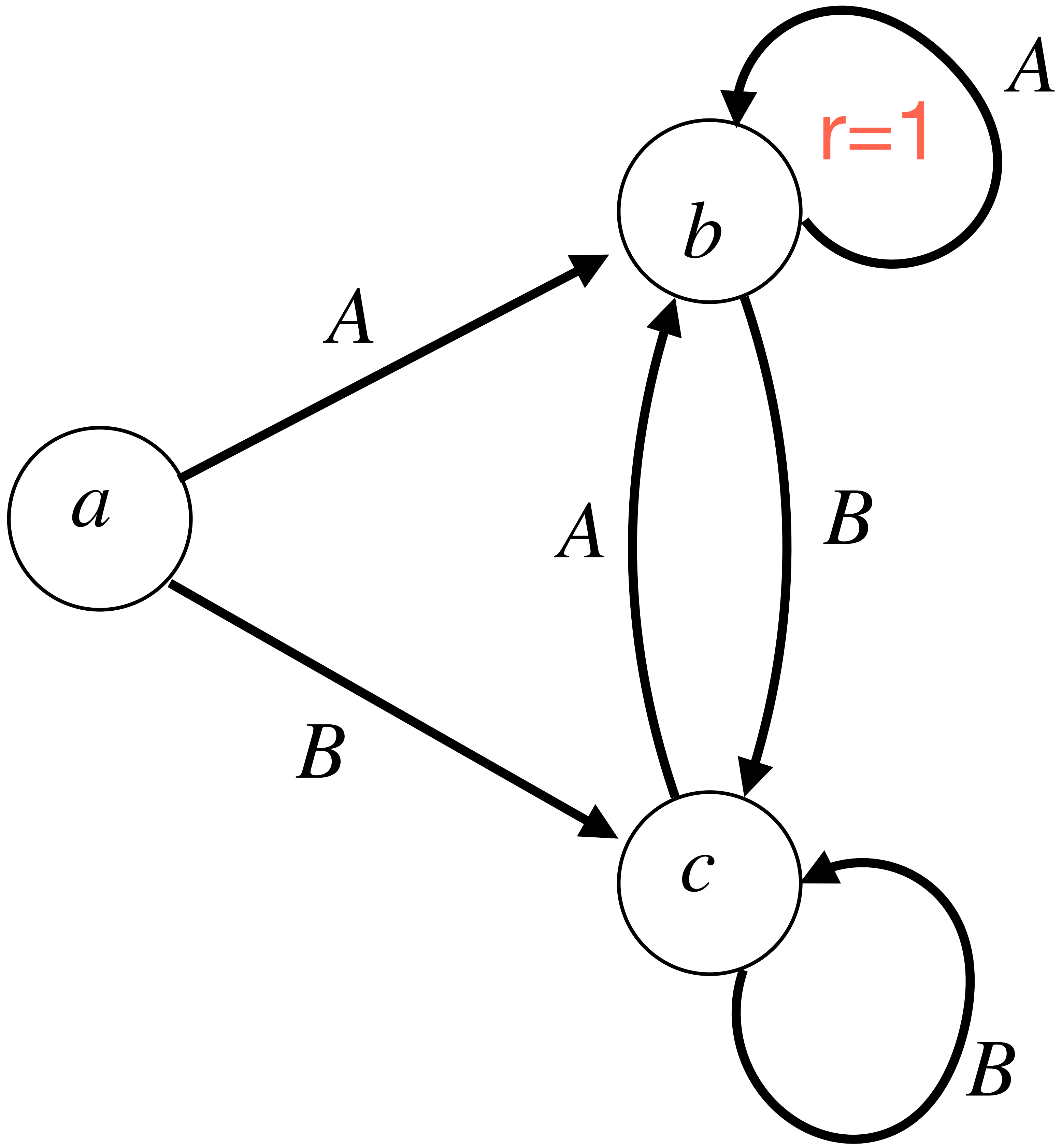

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- By the Markov property:

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$$= \mathbb{E}\left[r(s_0, a_0) + \gamma V^\pi(s_1) \middle| s_h = s \right]$$

$$= r(s, \pi(s)) + \gamma \sum_{s'} P(s' | s, \pi(s)) \; V^\pi(s')$$

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$V = r + \gamma P V$ (P: $|S| \times |S|$)

- What is the time complexity?

$O(|S|^3)$

Example of Optimal Policy $\pi^\star$, discounted case

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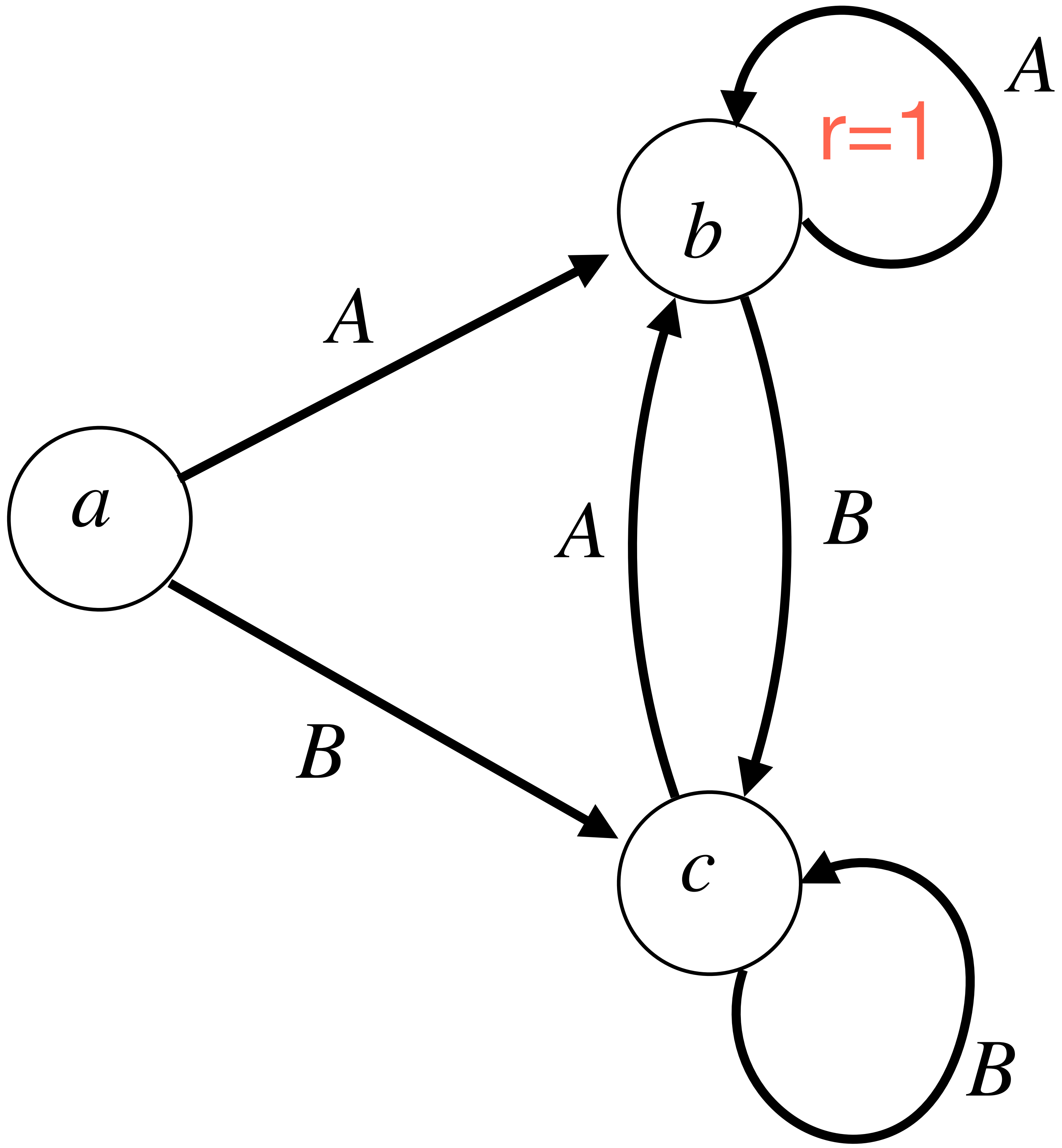

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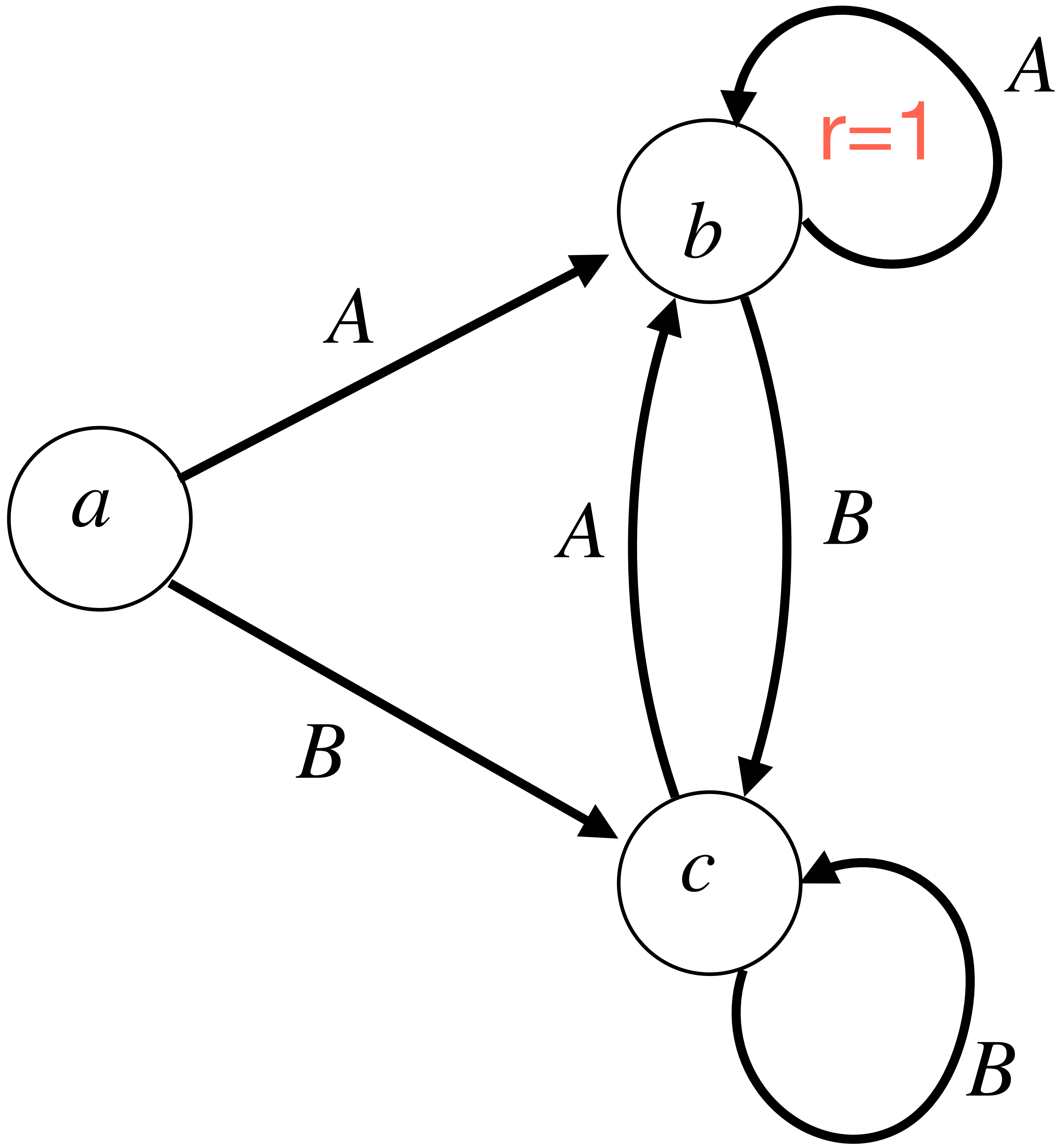

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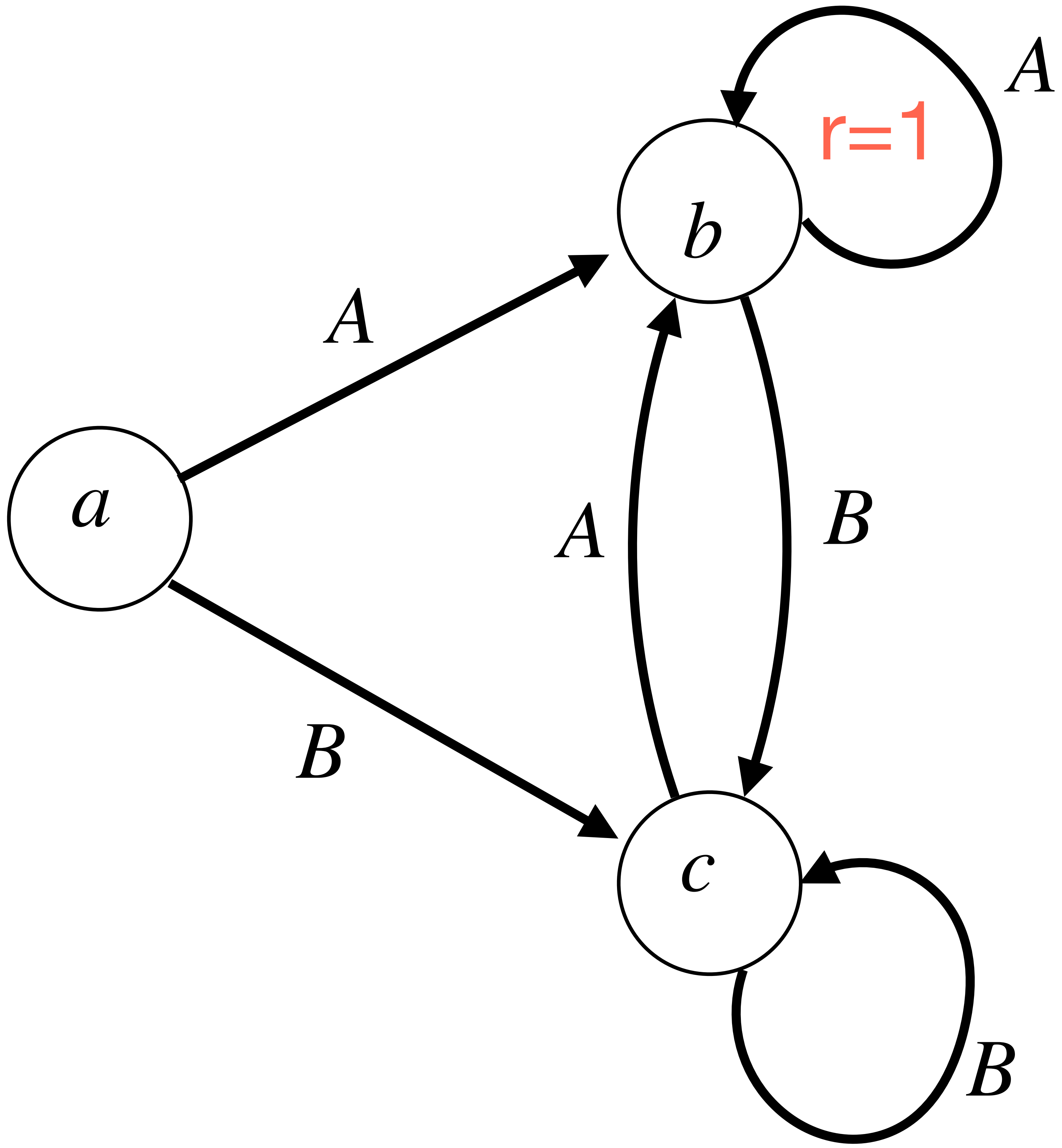

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- What's the optimal policy?
  $\pi^\star(s) = A, \forall s$
- What is optimal value function, $V^{\pi^\star} = V^\star$?

$$V^\star(a) = \frac{\gamma}{1-\gamma},\ V^\star(b) = \frac{1}{1-\gamma},\ V^\star(c) = \frac{\gamma}{1-\gamma}$$

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- Suppose $|S|$ states, $|A|$ actions.
  How many different stationary polices are there? $|A|^{|S|}$

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(again Π is the set of all time dependent, history dependent, stochastic policies)

- $\implies$ we can write: $V^\star = V^{\pi^\star}$ and $Q^\star = Q^{\pi^\star}$.

Summary:

- **Dynamic Programming lets us efficiently compute optimal policies.**
  - We remember the results on “sub-problems”
  - Optimal policies are history independent.
- Discounted infinite horizon MDP analogous to finite-horizon case

Attendance:

bit.ly/3RcTC9T

Feedback:

bit.ly/3RHtlxy