Markov Decision Processes & Dynamic Programming

Lucas Janson

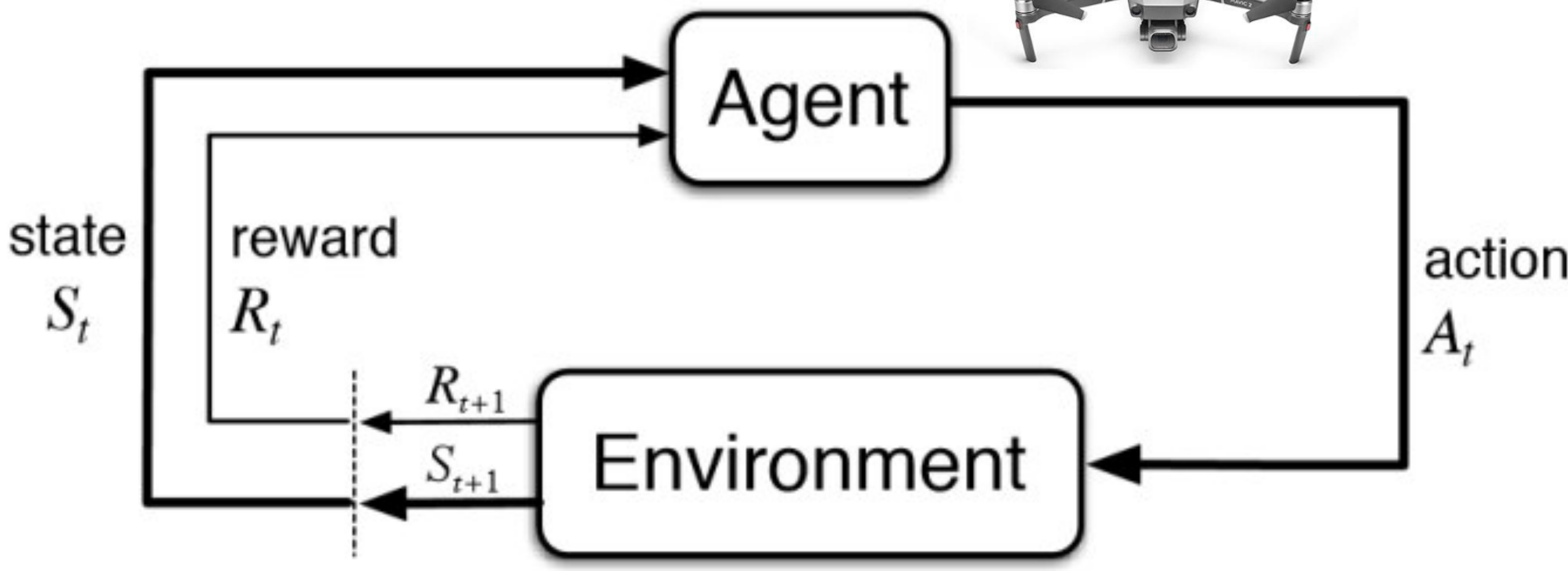
CS/Stat 184(0): Introduction to Reinforcement Learning

Fall 2024

Today

- Recap
- Problem Statement
- Bellman Consistency & Policy Evaluation
- Optimality
- The Bellman Equations & Dynamic Programming

Finite Horizon Markov Decision Processes (MDPs):



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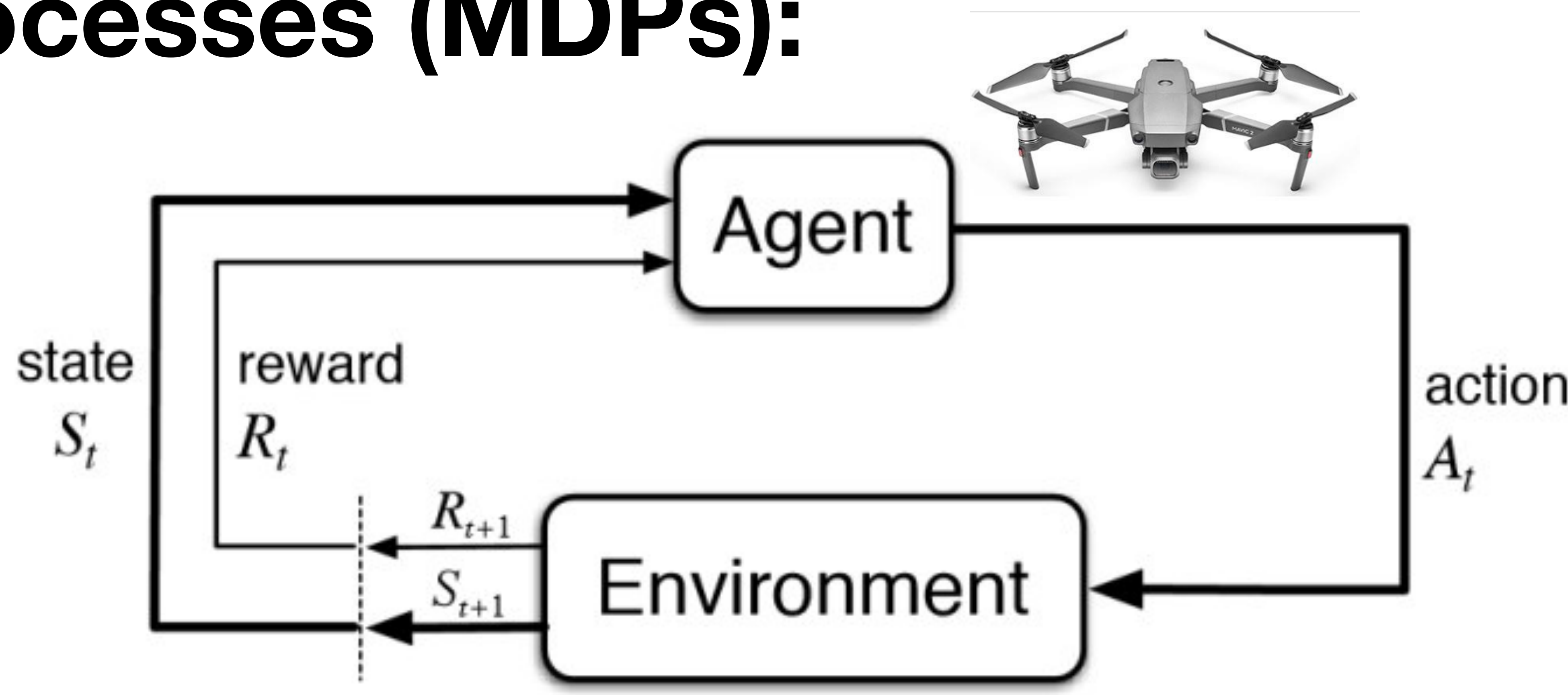

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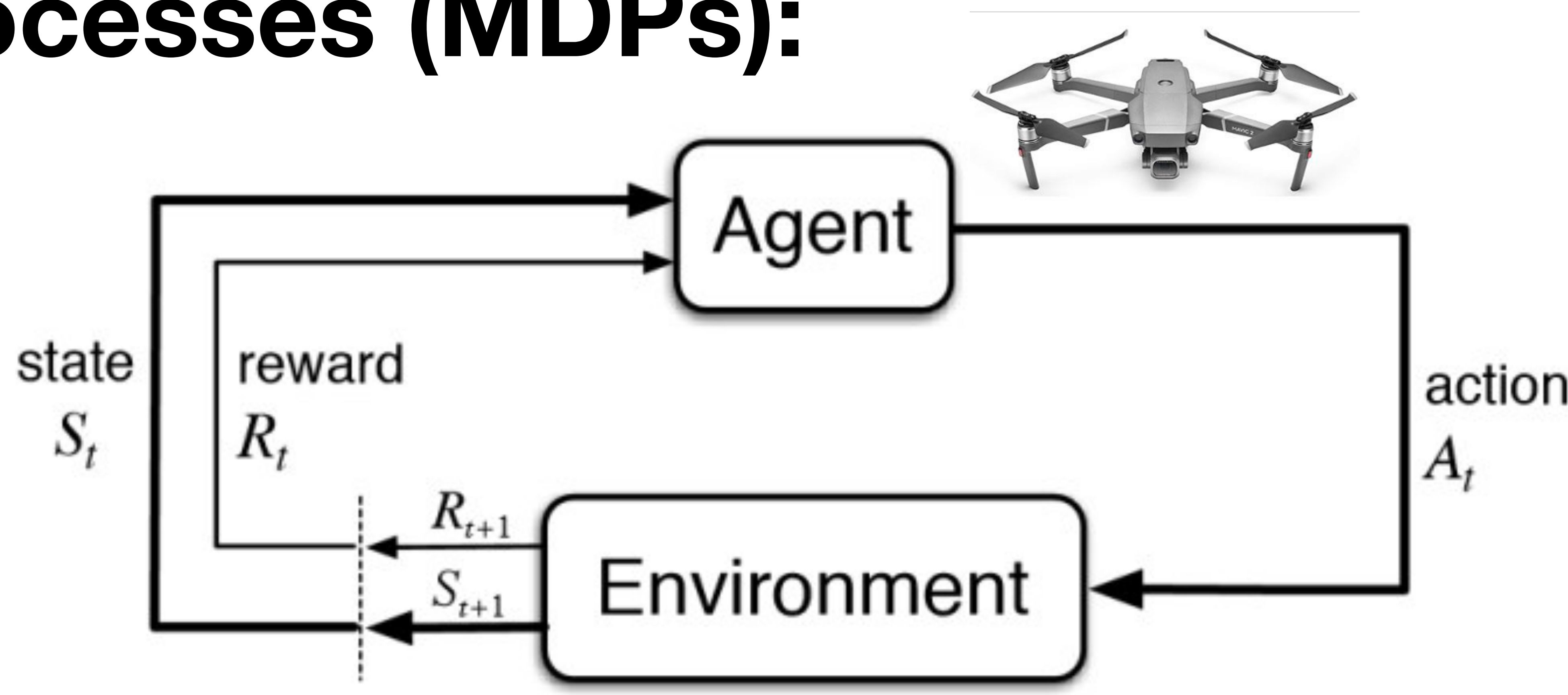

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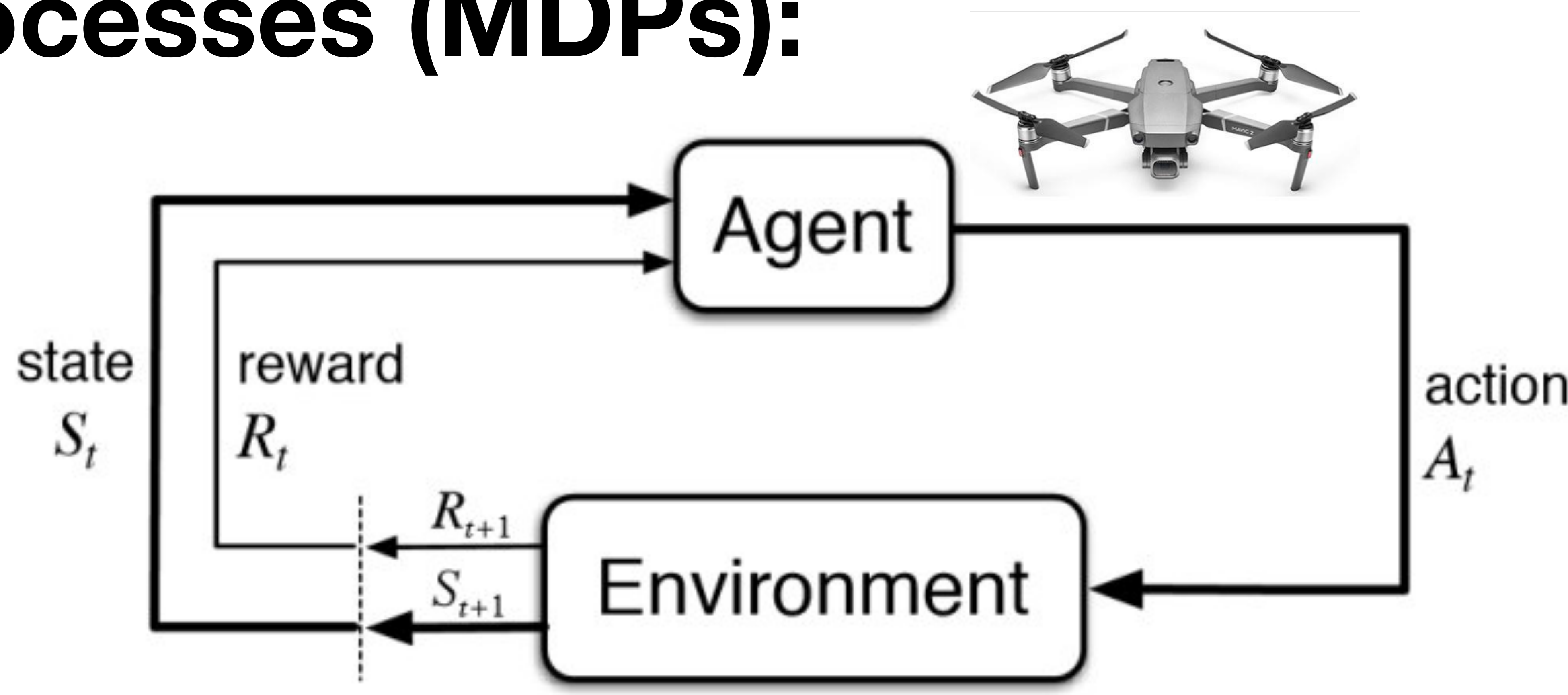

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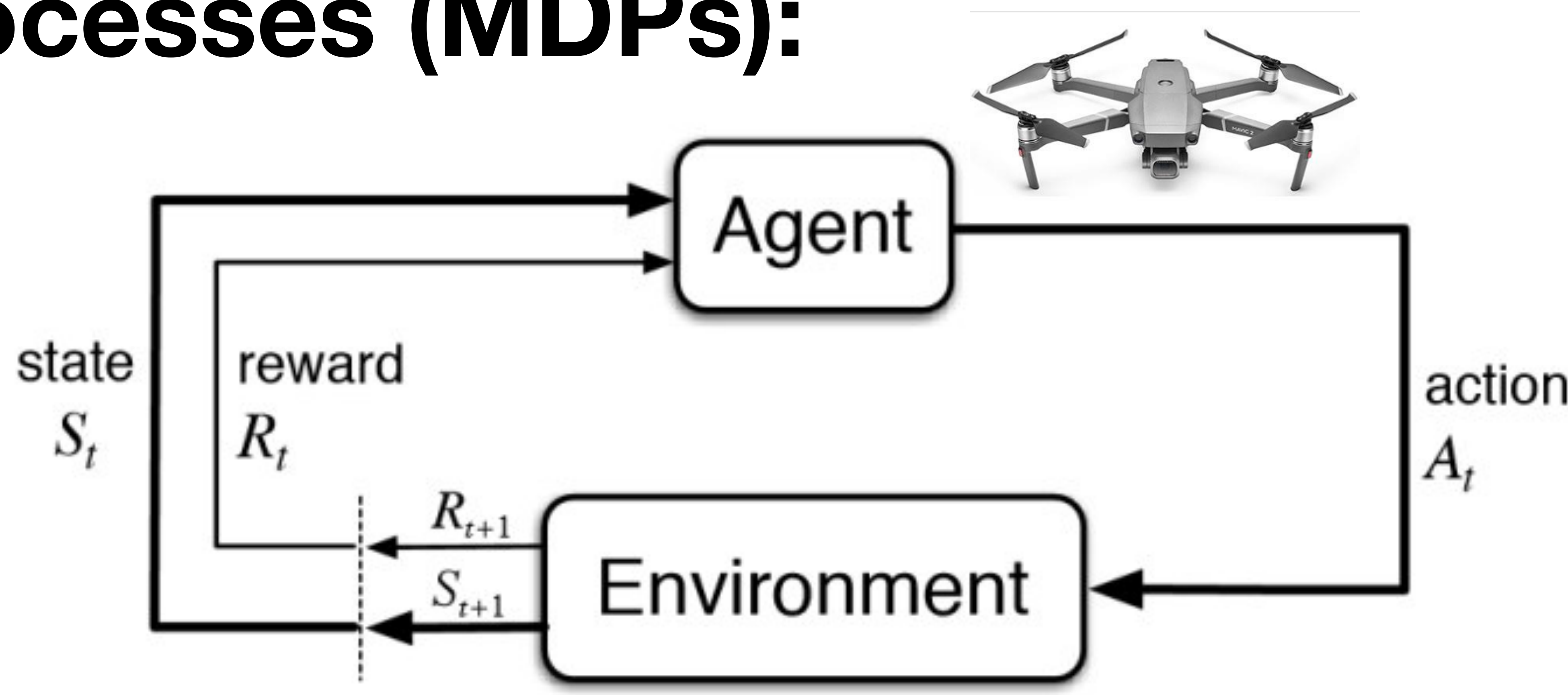

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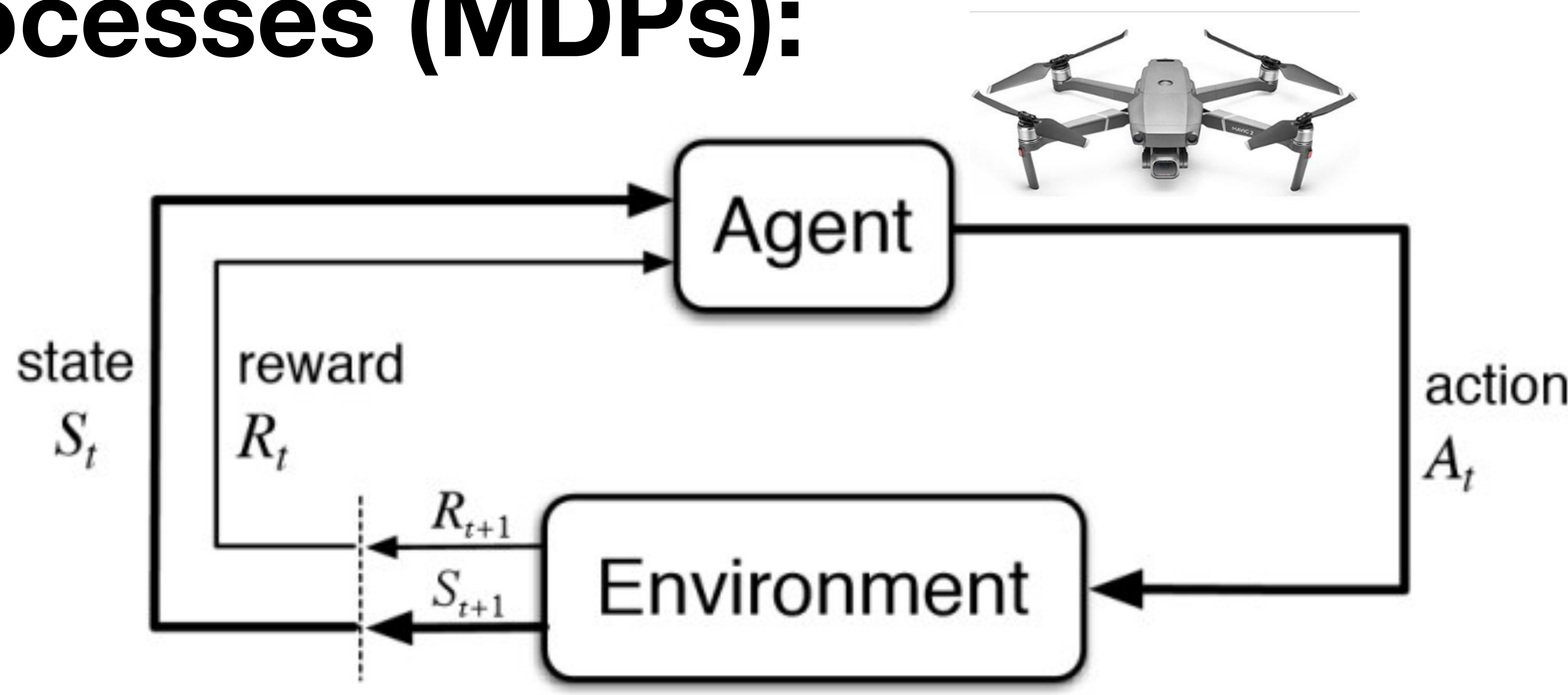

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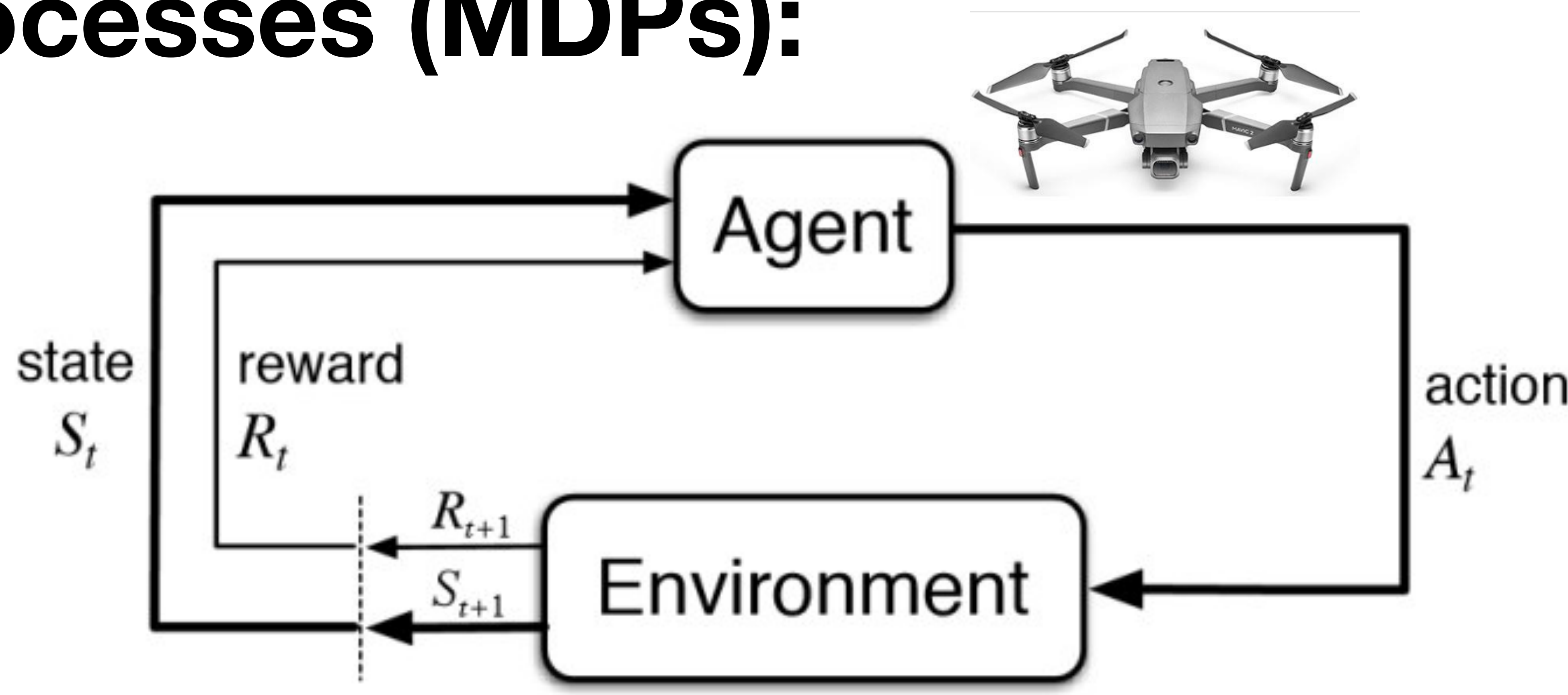

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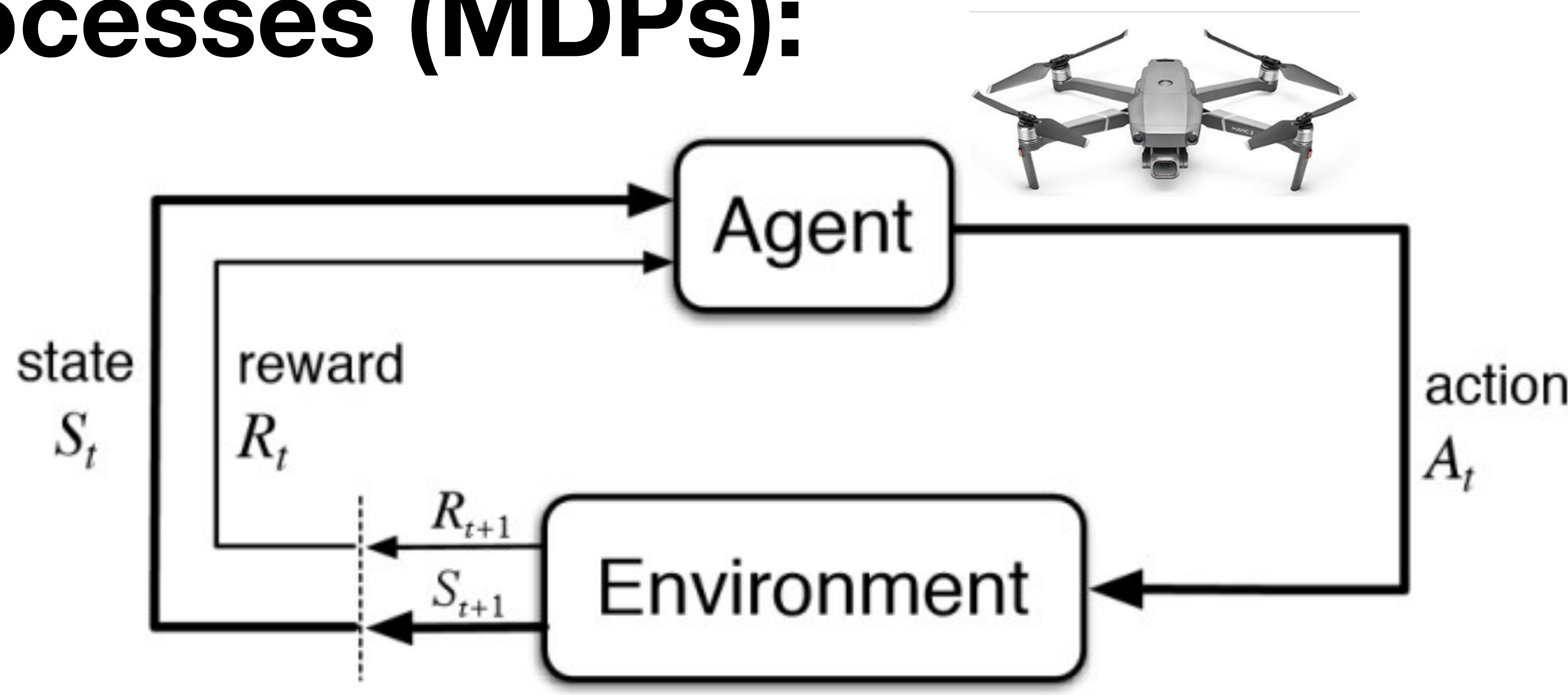

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  - A time horizon $H \in \mathbb{N}$

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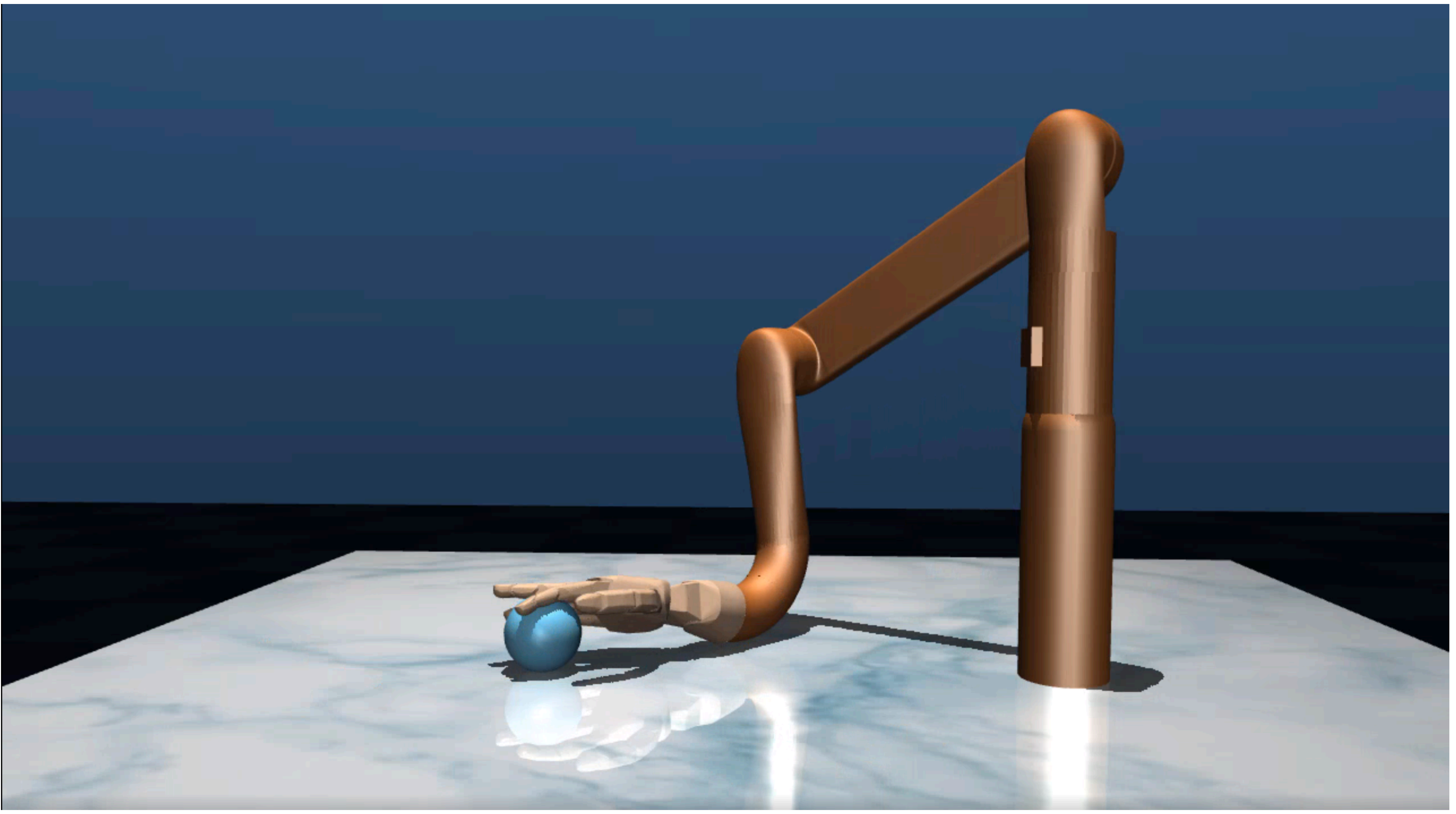
robot hand needs to pick the ball and hold it in a goal (x,y,z) position



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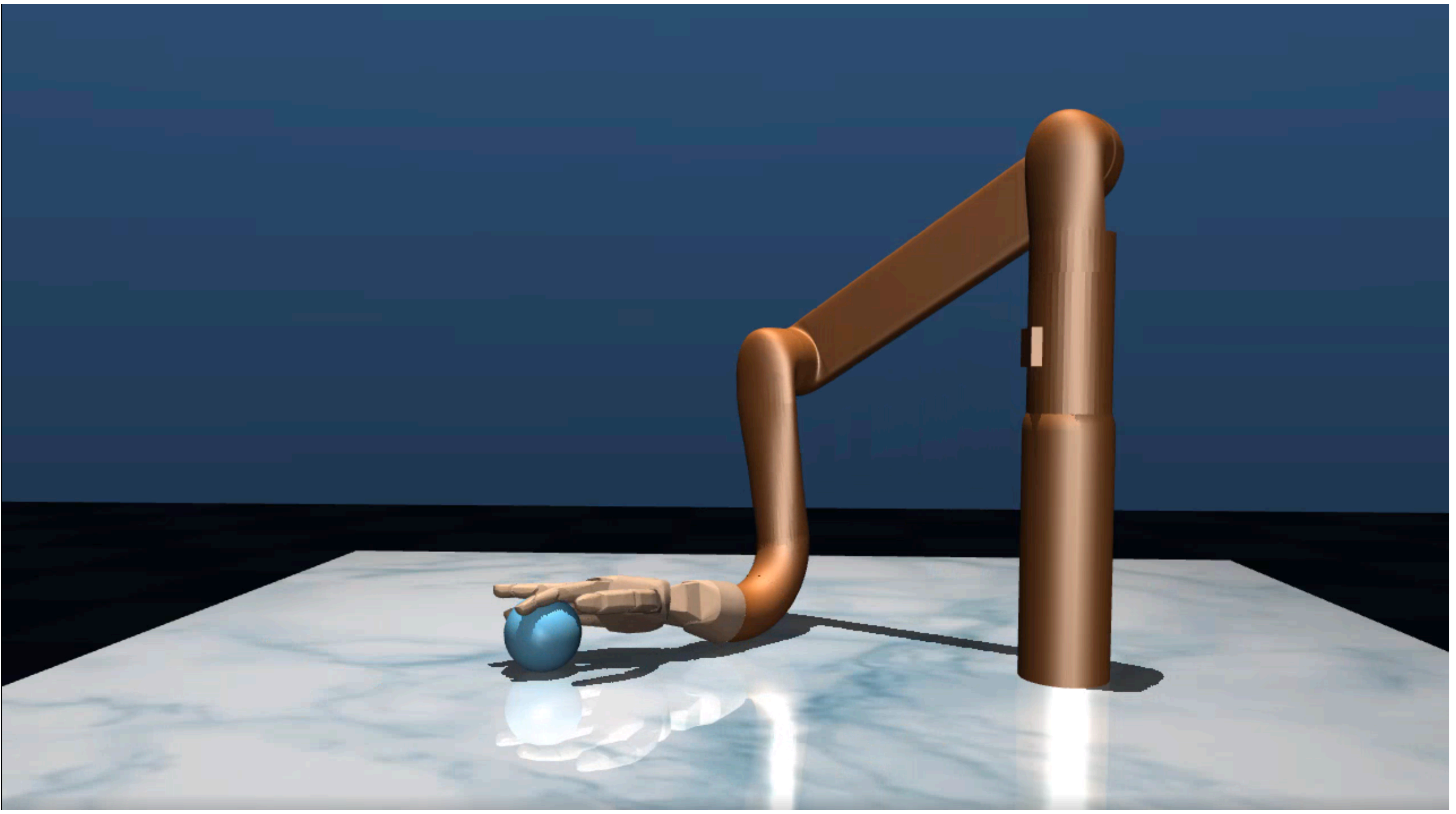

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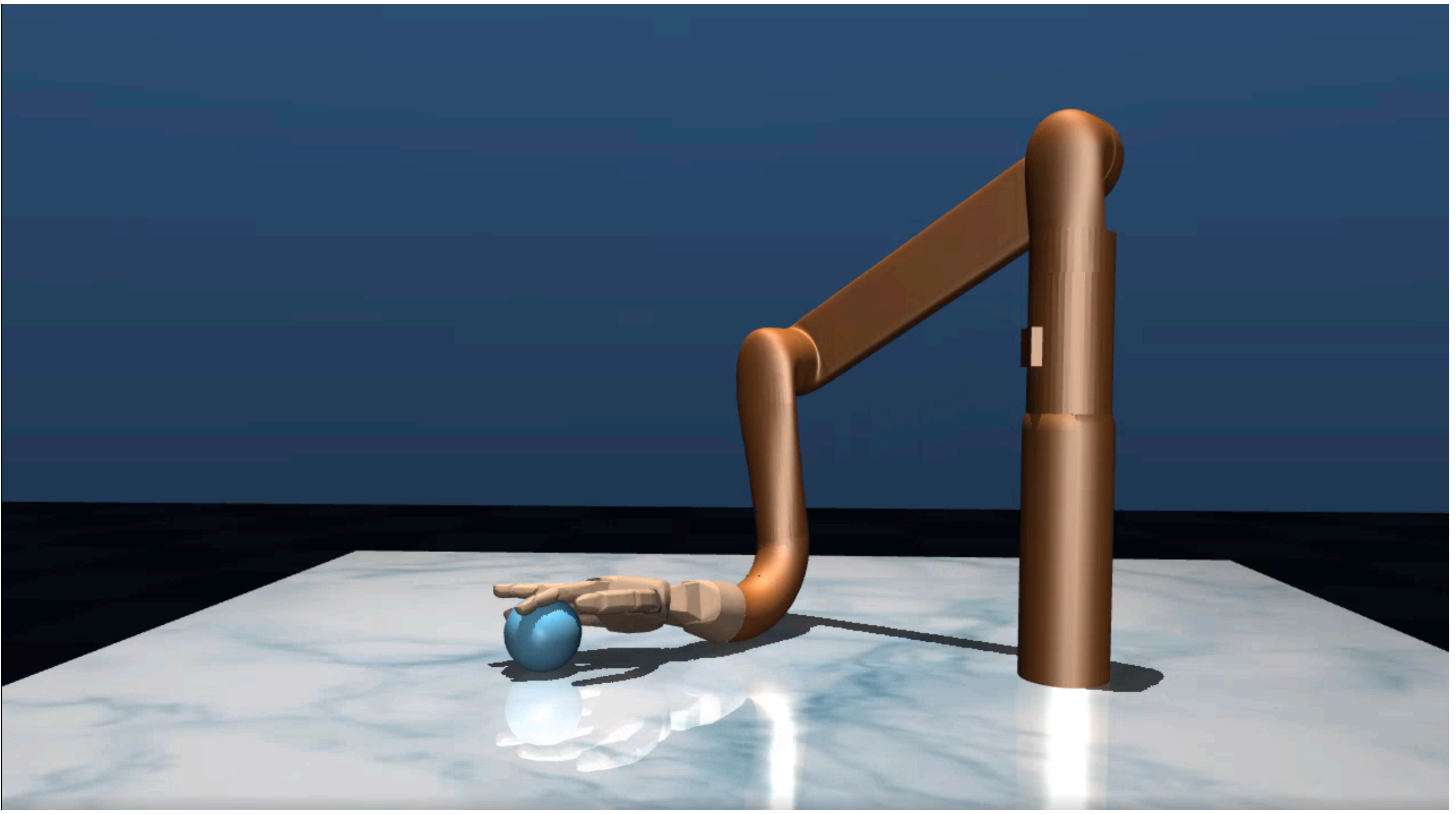

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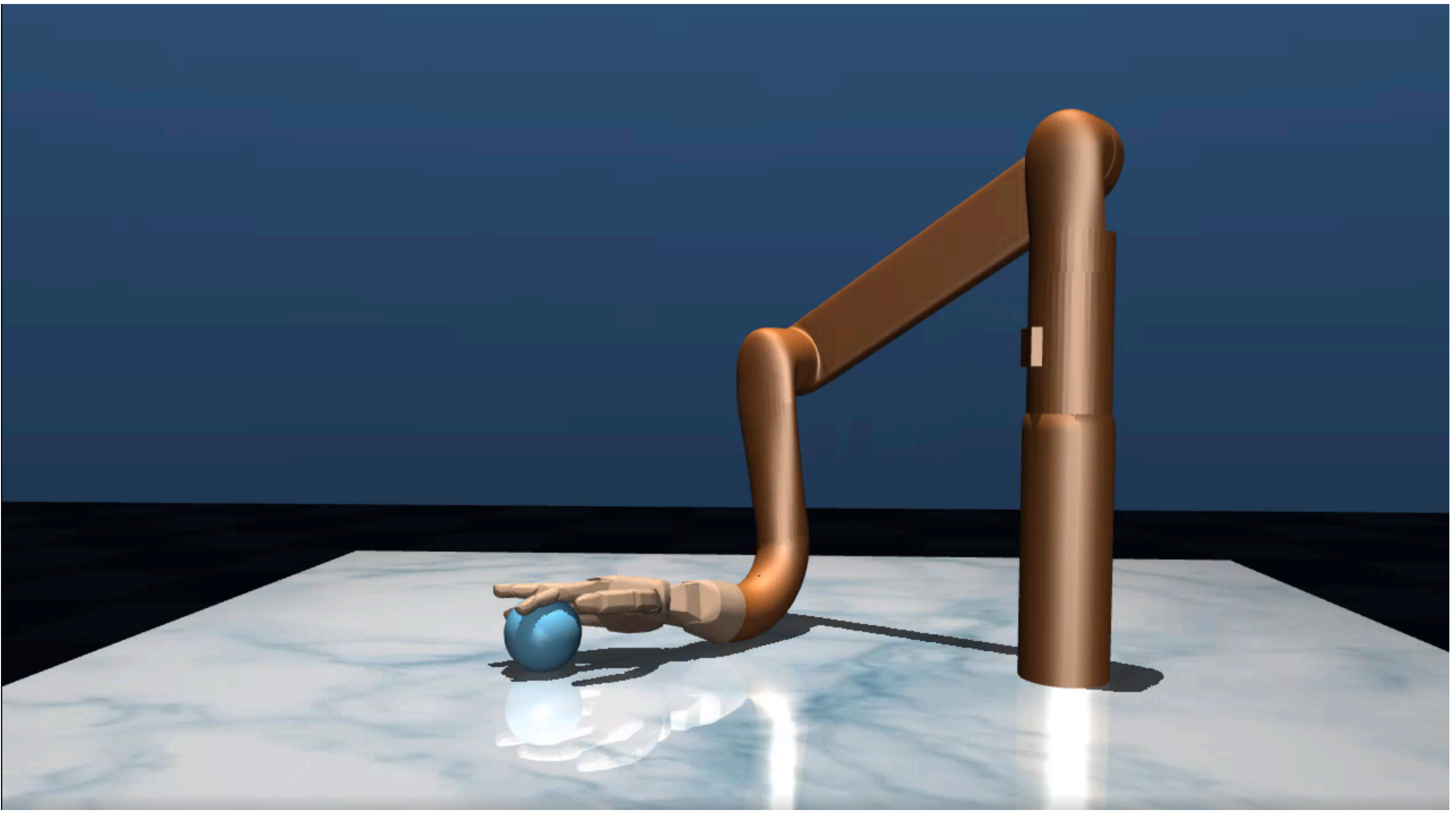

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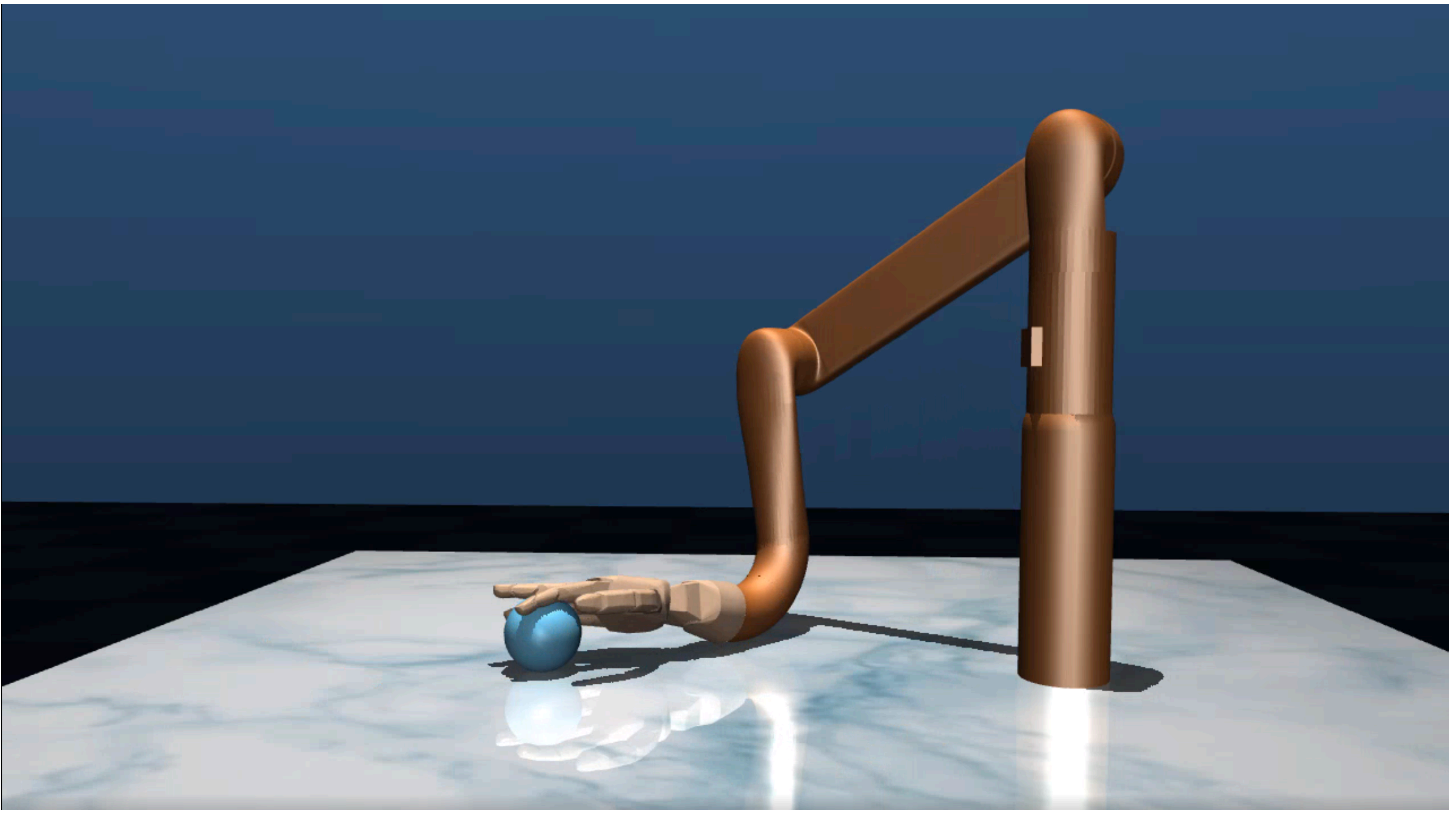

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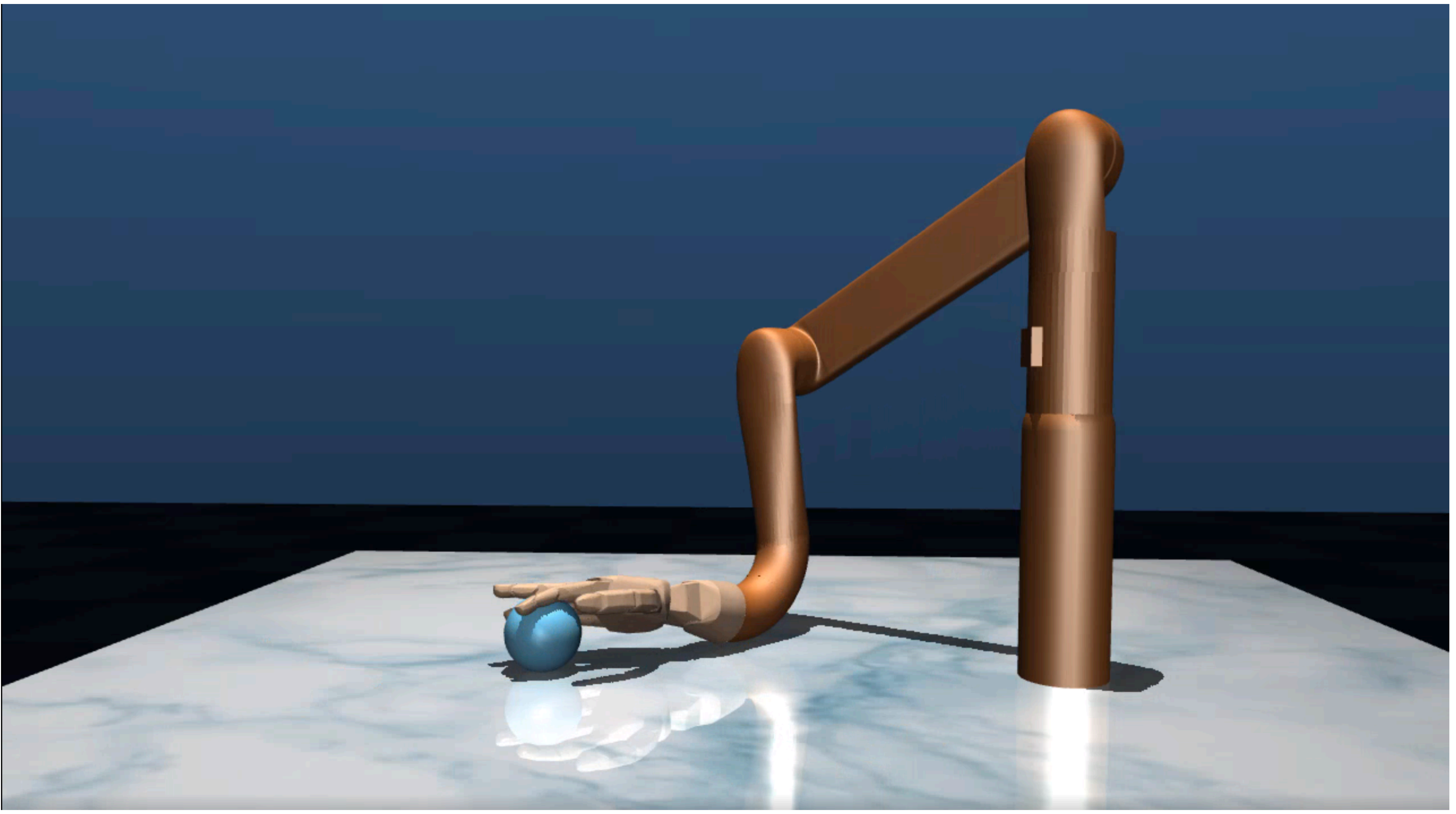

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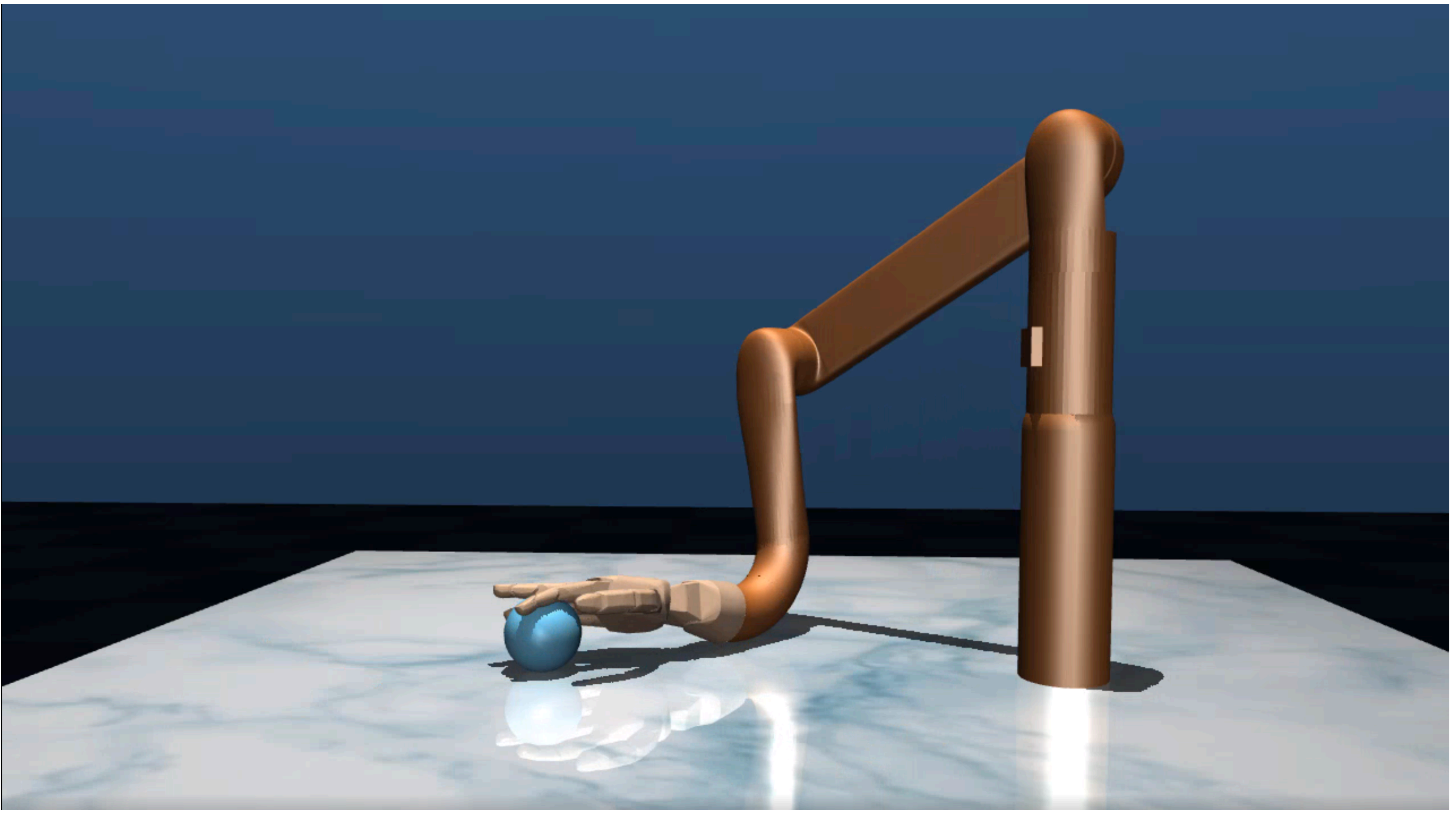

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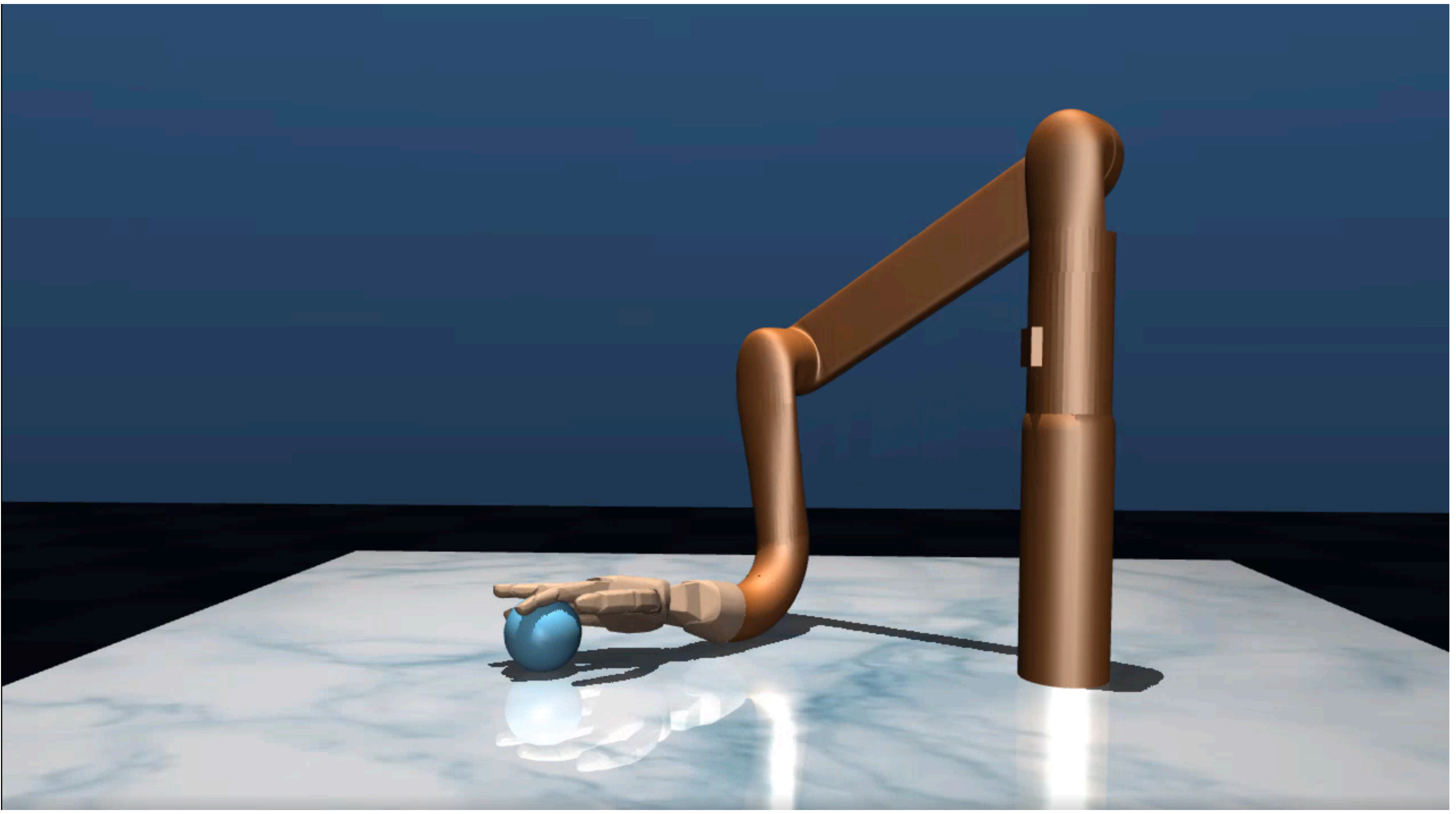

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$$\pi^\star = \arg\min_\pi \mathbb{E}\left[c(s_0, a_0) + c(s_1, a_1) + c(s_2, a_2) + \ldots c(s_{H-1}, a_{H-1}) \,\middle|\, s_0, \pi\right]$$

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  - The sampled trajectory is $\tau = \{s_0, a_0, r_0, s_1, a_1, r_1, \ldots, s_{H-1}, a_{H-1}, r_{H-1}\}$

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- Objective: find policy π that maximizes our expected cumulative episodic reward:

$$\max_\pi \mathbb{E}_{\tau \sim \rho_\pi}\left[r(s_0, a_0) + r(s_1, a_1) + \ldots + r(s_{H-1}, a_{H-1})\right]$$

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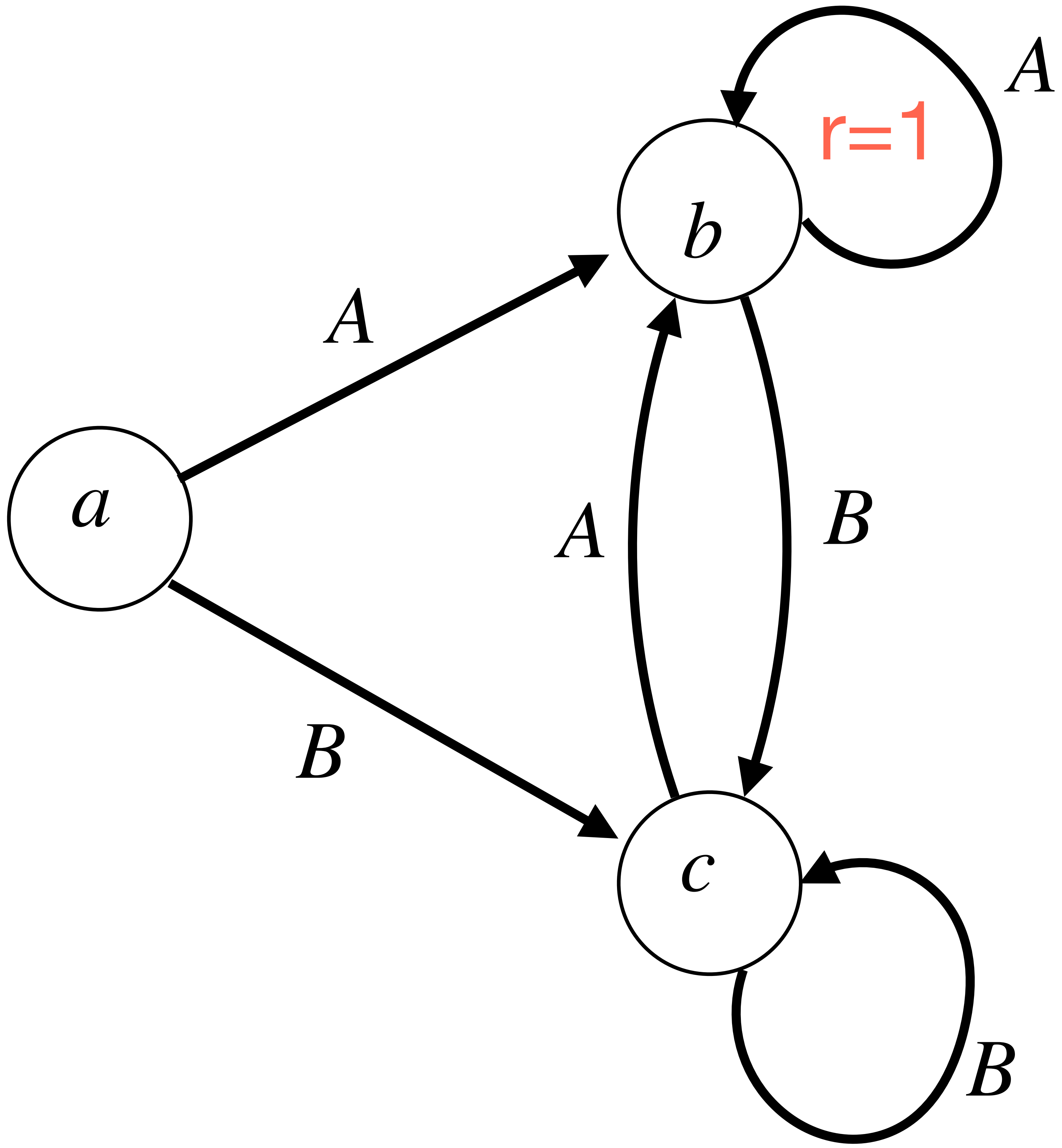
- At the last stage, for a stochastic policy:

for det policy, $V_{H-1}^\pi(s) = r(s, \pi_{H-1}(s))$

$$Q_{H-1}^\pi(s, a) = r(s, a) \qquad V_{H-1}^\pi(s) = \sum_a \pi_{H-1}(a \mid s) r(s, a)$$

Example of Policy Evaluation (i.e. computing V^{π} and Q^{π})

Consider the following **deterministic** MDP w/ 3 states & 2 actions, with $H = 3$



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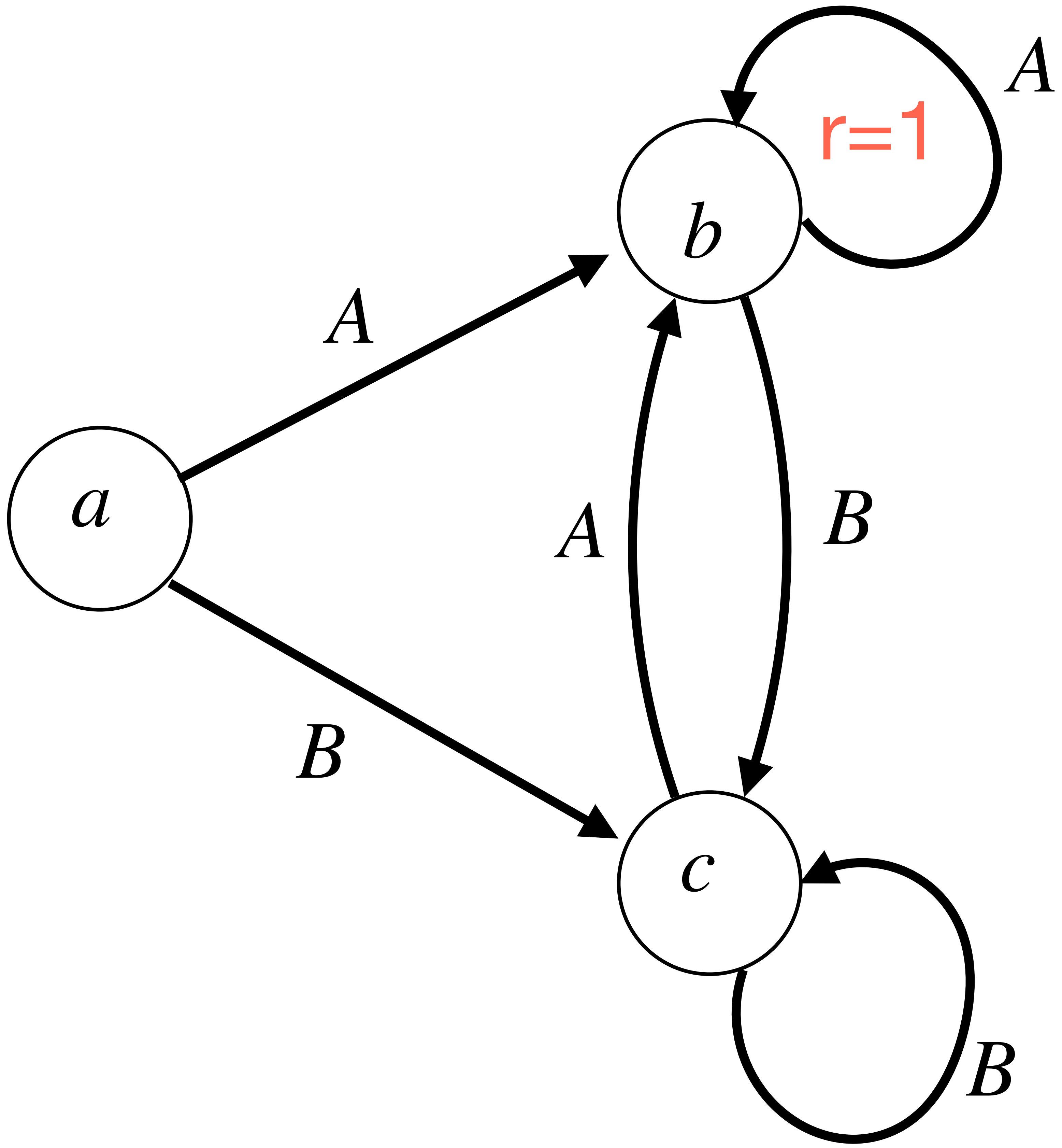

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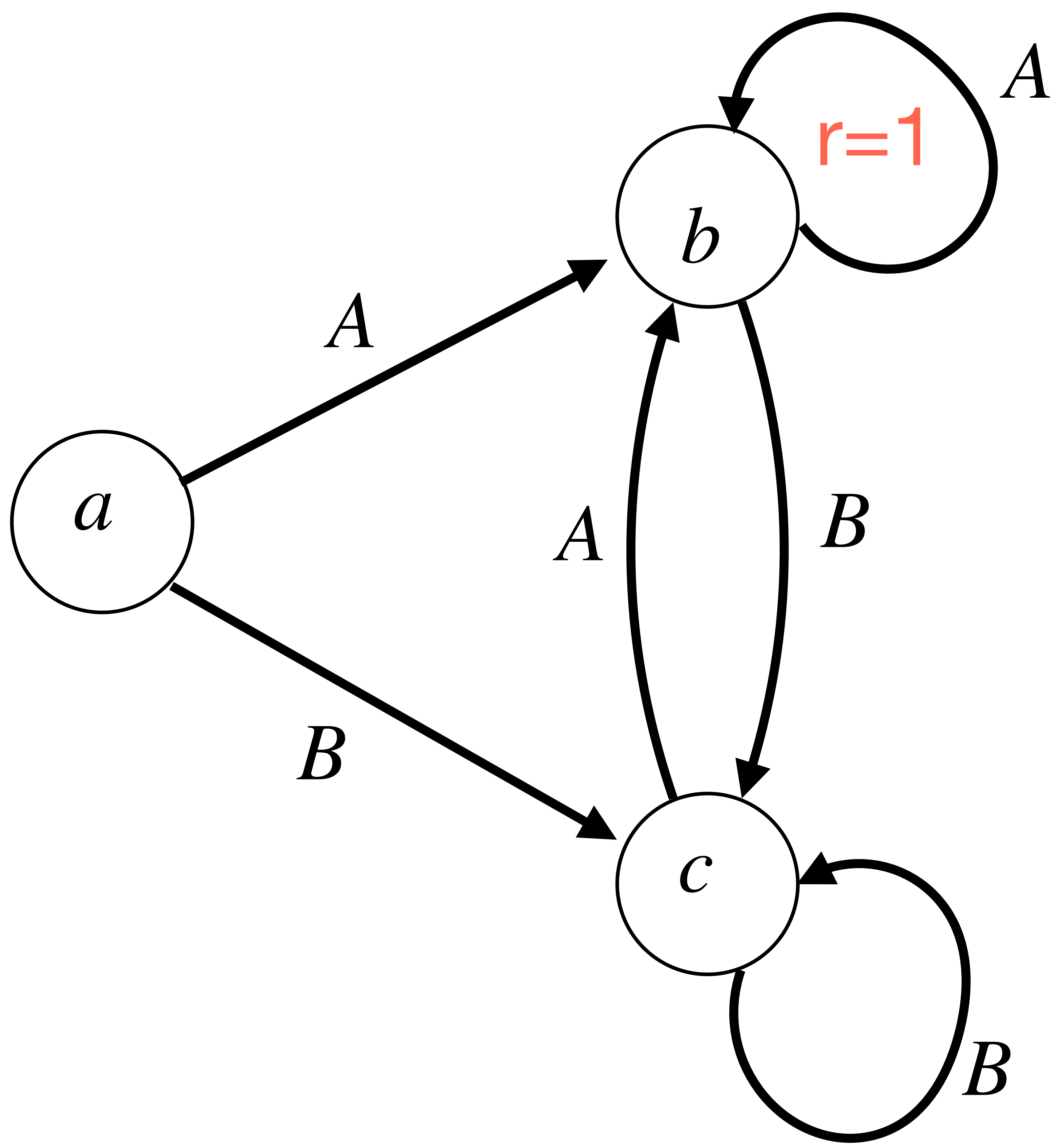

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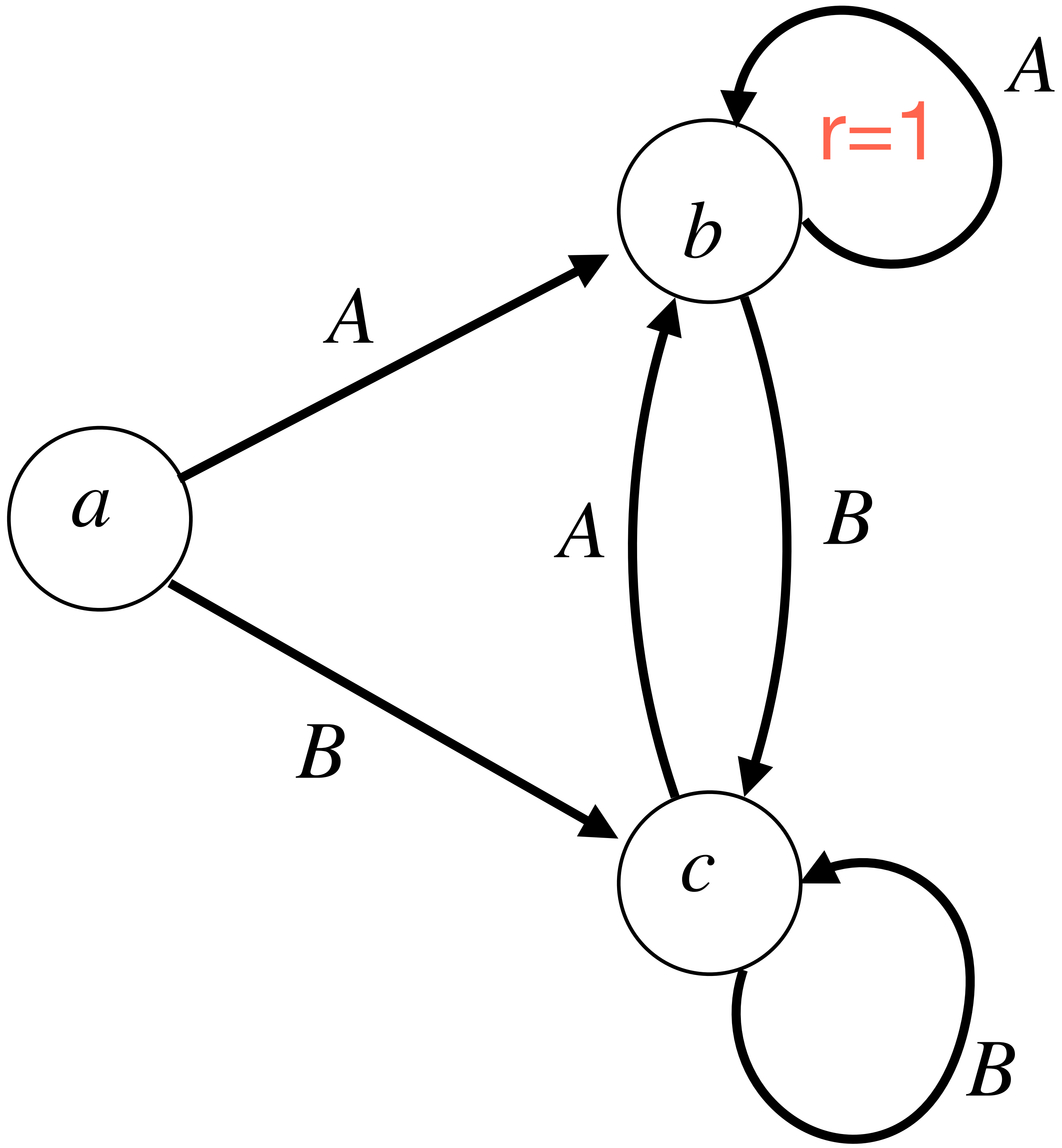

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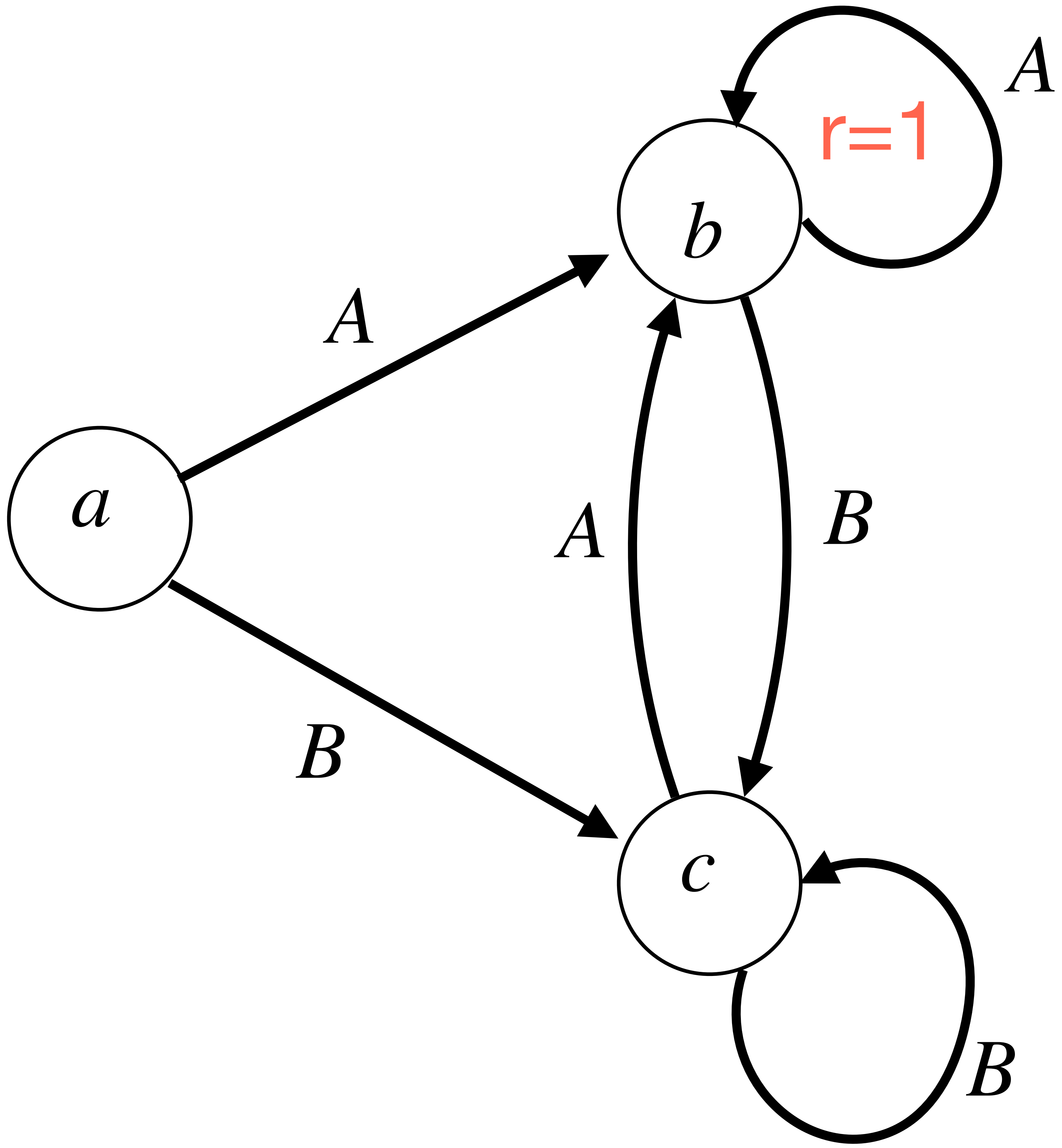

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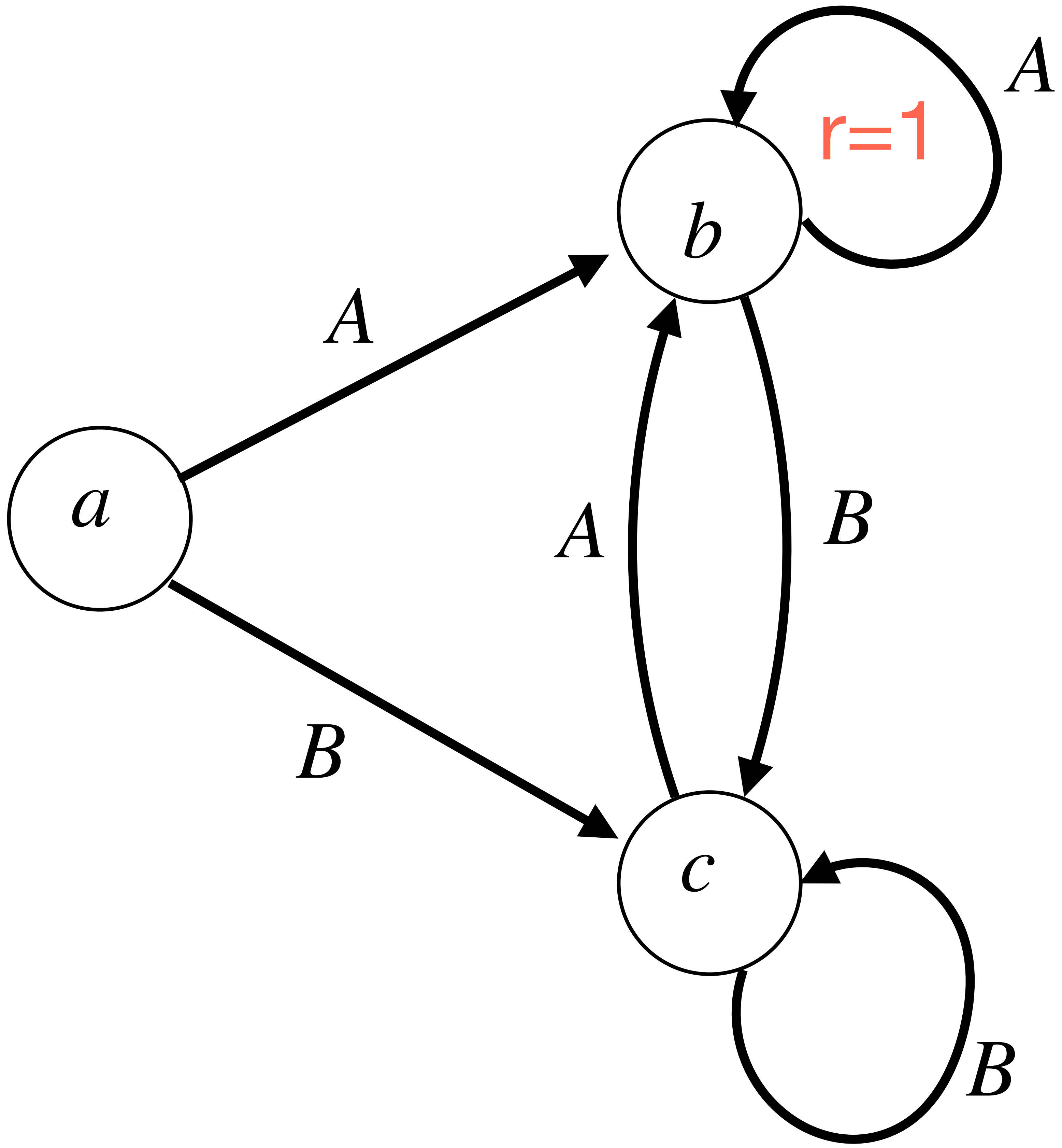

Reward: $r(b, A) = 1$, & 0 everywhere else

- Consider the deterministic policy
  $\pi_0(s) = A, \;\; \pi_1(s) = A, \;\; \pi_2(s) = B, \;\; \forall s$

- What is V^π?
  $V_2^\pi(a) = 0, \;\; V_2^\pi(b) = 0, \;\; V_2^\pi(c) = 0$

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Bellman Consistency

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- What is the total computational complexity of DP? $O(H|S|^2)$

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- Problem Statement
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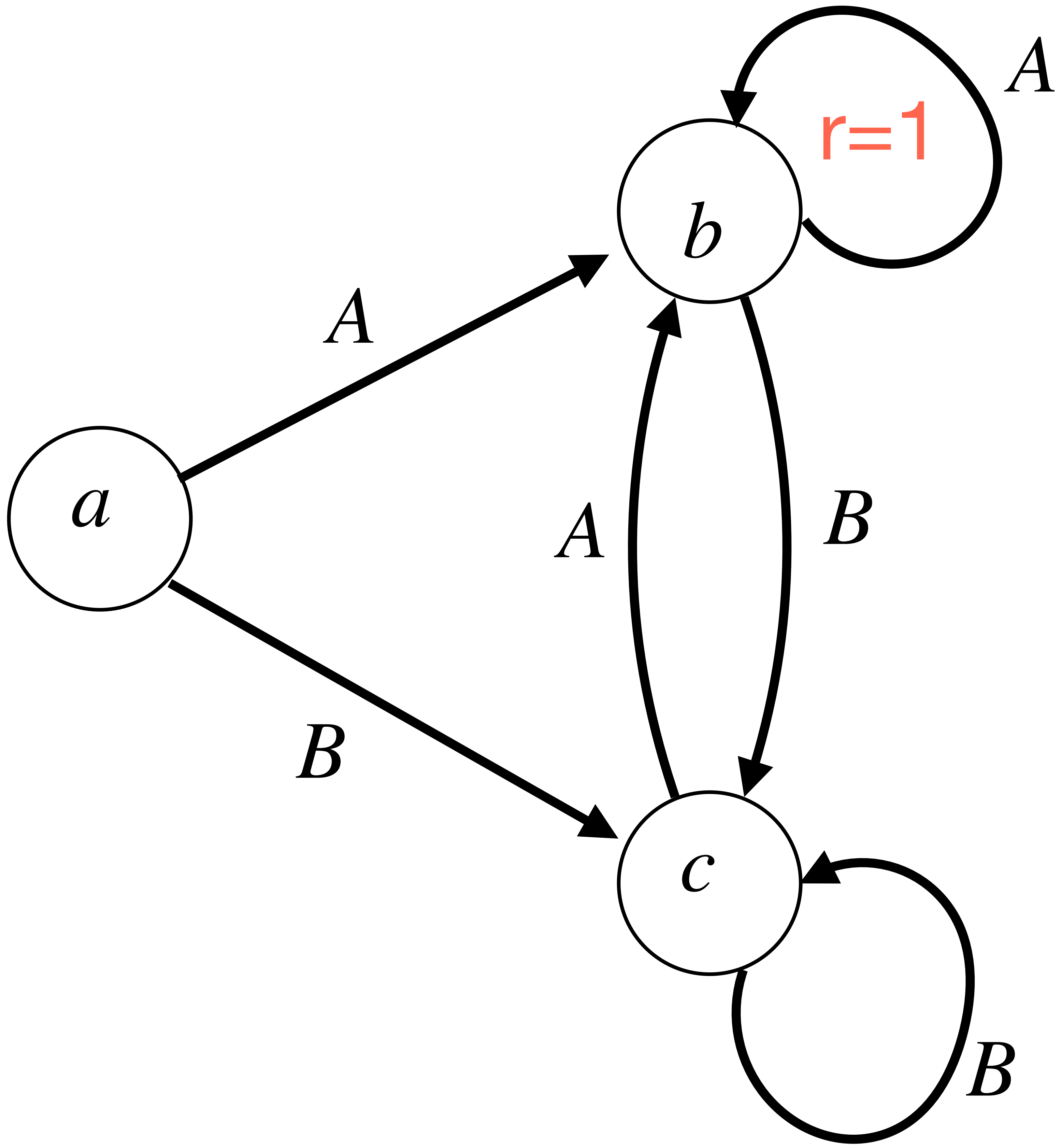

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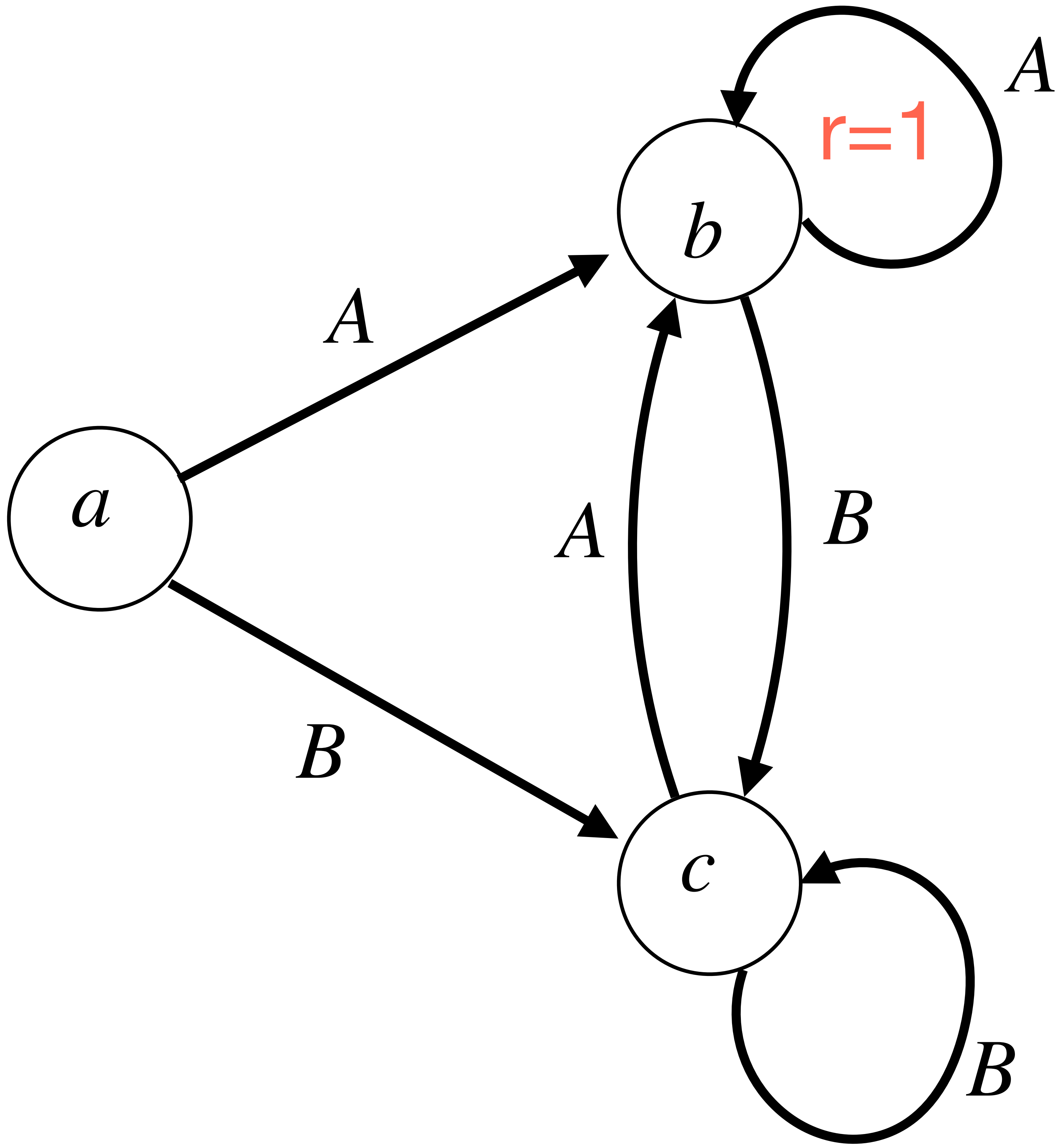

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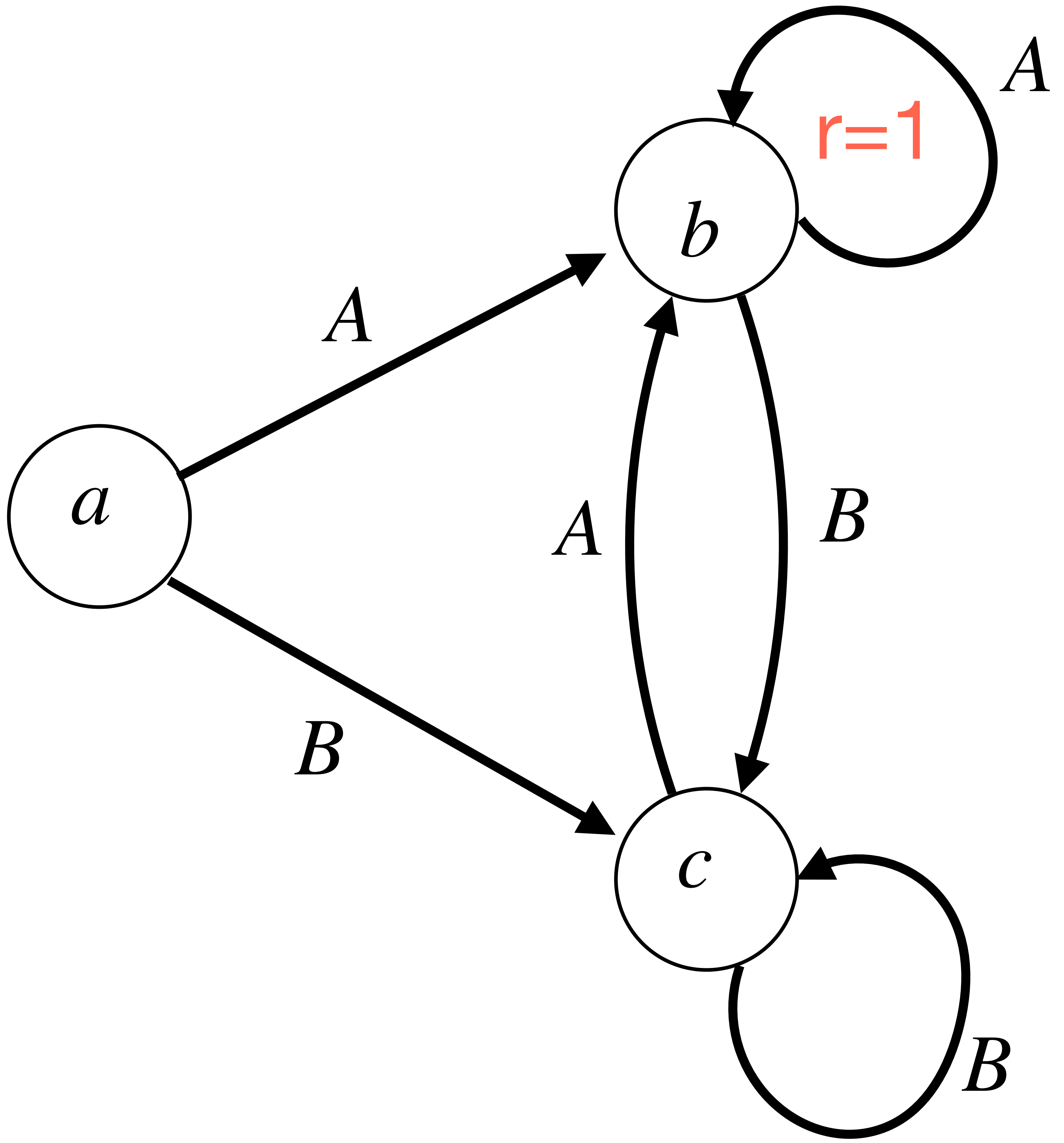

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- What's the optimal policy?
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- What is optimal value function, $V^{\pi^\star} = V^\star$?
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  How many different polices there are?
- Can we do better?

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- $\implies$ we can write: $V_h^\star = V_h^{\pi^\star}$ and $Q_h^\star = Q_h^{\pi^\star}$.
- $\implies$ $\pi^\star$ doesn't depend on the initial state distribution μ.

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  - "Only the state matters": how got here doesn't matter to where we go next, conditioned on the action.
    - This explains both determinism and history-independence
- Caveat: some legitimate reward functions are not additive/linear (so, naively, not an MDP). (But, RL is general: think about redefining the state so you can do these.)

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  - The optimal policy is: $\pi_h^\star(s) = \arg\max_a \left\{ r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V_{h+1}^\star(s') \right] \right\}$.

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- What is the total computational complexity of DP?

Summary:

- **Dynamic Programming lets us efficiently compute optimal policies.**
  - We remember the results on “sub-problems”
  - Optimal policies are history independent.

Attendance:
bit.ly/3RcTC9T

Feedback:
bit.ly/3RHtlxy