Contextual Bandits

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CS/Stat 184(0): Introduction to Reinforcement Learning

Fall 2024

Today

- Feedback from last lecture
- Recap
- UCB-VI for linear MDPs
- Recall: Contextual Bandits
- LinUCB

Feedback from feedback forms

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1. Thank you to everyone who filled out the forms!

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Exploration in MDP: make it a bandit and do UCB?

Q: given a discrete MDP, how many unique deterministic policies are there?

$$\left(|A|^{|S|}\right)^H$$

So treating each policy as an "arm" and running UCB gives us regret $\tilde{O}(\sqrt{|A|^{|S|H}N})$

This seems bad, so are MDPs just super hard or can we do better?

Tabular UCB-VI

For $n = 1 \to N$:

1. Set $N_h^n(s, a) = \sum_{i=1}^{n-1} \mathbf{1}\{(s_h^i, a_h^i) = (s, a)\}, \forall s, a, h$

2. Set $N_h^n(s, a, s') = \sum_{i=1}^{n-1} \mathbf{1}\{(s_h^i, a_h^i, s_{h+1}^i) = (s, a, s')\}, \forall s, a, a', h$

3. Estimate $\hat{P}^n$: $\hat{P}_h^n(s' | s, a) = \frac{N_h^n(s, a, s')}{N_h^n(s, a)}, \forall s, a, s', h$

4. Plan: $\pi^n =$ VI $\left(\{\hat{P}_h^n, r_h + b_h^n\}_h \right)$, with $b_h^n(s, a) = cH\sqrt{\frac{\log(|S||A|HN/\delta)}{N_h^n(s, a)}}$

5. Execute π^n : $\{s_0^n, a_0^n, r_0^n, \ldots, s_{H-1}^n, a_{H-1}^n, r_{H-1}^n, s_H^n\}$

High-level Idea: Exploration Exploitation Tradeoff

Upper bound per-episode regret: $V_0^\star(s_0) - V_0^{\pi^n}(s_0) \leq \hat{V}_0^n(s_0) - V_0^{\pi^n}(s_0)$ by construction of b_h^n

1. What if $\hat{V}_0^n(s_0) - V_0^{\pi^n}(s_0)$ is small?

Then π^n is close to $\pi^\star$, i.e., we are doing exploitation

2. What if $\hat{V}_0^n(s_0) - V_0^{\pi^n}(s_0)$ is large?

Some $b_h^n(s,a)$ must be large (or some $\hat{P}_h^n(\cdot \mid s,a)$ estimation errors must be large, but with high probability any $\hat{P}_h^n(\cdot \mid s,a)$ with high error must have small $N_h^n(s,a)$ and hence high $b_h^n(s,a)$)

Large $b_h^n(s,a)$ means π^n is being encouraged to do (s,a), since it will apparently have very high reward, i.e., exploration

$$\mathbb{E}\left[\text{Regret}_N\right] := \mathbb{E}\left[\sum_{n=1}^{N}\left(V^\star - V^{\pi^n}\right)\right] \leq \widetilde{O}\left(H^2\sqrt{|S||A|N}\right)$$

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Linear MDP Definition

Finite horizon time-dependent episodic MDP $\mathscr{M} = \{S, A, H, \{r\}_h, \{P\}_h, s_0\}$

S & A could be large or even continuous, hence poly$(|S|, |A|)$ is not acceptable

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$$P_h(s'|s,a) = \mu_h^\star(s') \cdot \phi(s,a), \quad \mu_h^\star : S \mapsto \mathbb{R}^d, \quad \phi : S \times A \mapsto \mathbb{R}^d$$

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Feature map ϕ is known to the learner!

(We assume reward is known, i.e., $\theta^\star$ is known)

Planning in Linear MDP: Value Iteration

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Indeed we can show that $Q_h^\pi(\cdot, \cdot)$
Is linear with respect to ϕ as well, for any π, h

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Penalized Linear Regression:

$$\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \lambda\|\mu\|_F^2$$

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How to choose $b_h^n(s, a)$?

Chebyshev-like approach, similar to in linUCB (will cover later this lecture):

$$b_h^n(s, a) = \beta\sqrt{\phi(s, a)^\top (A_h^n)^{-1} \phi(s, a)}, \quad \beta = \widetilde{O}(dH)$$

linUCB-VI: Put All Together

For $n = 1 \to N$:

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We have K many arms; label them $1, \ldots, K$

Each arm has an unknown reward distribution, i.e., $\nu_k \in \Delta([0,1])$,

w/ mean $\mu_k = \mathbb{E}_{r \sim \nu_k}[r]$

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Which user comes in next is random, but we have some context to tell situations apart and hence learn different optimal actions

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If we knew everything about the environment, we'd want to use the optimal policy

$$\pi^\star(x_t) := \arg\max_{k \in \{1,\ldots,K\}} \mu^{(k)}(x_t), \qquad \text{where } \mu^{(k)}(x) := \mathbb{E}_{r \sim \nu^{(k)}(x)}[r]$$

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Context at time t encoded into a variable x_t that we see before choosing our action

x_t is drawn i.i.d. at each time point from a distribution ν_x on sample space $\mathcal{X}$

x_t then affects the reward distributions of each arm, i.e., if we choose arm k, we get a reward that is drawn from a distribution that depends on x_t, namely, $\nu^{(k)}(x_t)$

Accordingly, we should also choose our action a_t in a way that depends on x_t, i.e., our action should be chosen by a function of x_t (a policy), namely, $\pi_t(x_t)$

If we knew everything about the environment, we'd want to use the optimal policy

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$\pi^\star$ is the policy we compare to in computing regret

Recall: Contextual bandit environment

Formally, a contextual bandit is the following interactive learning process:

For $t = 0 \rightarrow T - 1$

1. Learner sees context $x_t \sim \nu_x$ Independent of any previous data
2. Learner pulls arm $a_t = \pi_t(x_t) \in \{1, \ldots, K\}$ π_t policy learned from all data seen so far
3. Learner observes reward $r_t \sim \nu^{(a_t)}(x_t)$ from arm a_t in context x_t

Note that if the context distribution ν_x always returns the same value (e.g., 0), then the contextual bandit reduces to the original multi-armed bandit

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UCB algorithm conceptually identical as long as $|\mathcal{X}|$ finite:

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Example: showing an ad on a NYT article on politics vs a NYT article on sports: Not *identical* readership, but still both on NYT, so probably still *similar* readership!

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Lower dimension makes learning easier, but model could be wrong/biased

Today

- Feedback from last lecture
- Recap
- UCB-VI for linear MDPs
- Recall: Contextual Bandits
- LinUCB

Linear model fitting

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proof: $\nabla_\theta \left[\sum_{\tau=0}^{t-1} (r_\tau - x_\tau^\top \theta)^2 1_{\{a_\tau = k\}} \right] = 2 \sum_{\tau=0}^{t-1} x_\tau (r_\tau - x_\tau^\top \theta) 1_{\{a_\tau = k\}} = 0 \quad \Rightarrow \quad \sum_{\tau=0}^{t-1} x_\tau r_\tau 1_{\{a_\tau = k\}} = \theta \sum_{\tau=0}^{t-1} x_\tau x_\tau^\top 1_{\{a_\tau = k\}}$

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$A_t^{(k)}$ must be invertible, which basically requires $N_t^{(k)} \geq d$

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Recall: $\hat{\theta}_t^{(k)} = \theta^{(k)} + (A_t^{(k)})^{-1} \sum_{\tau=0}^{t-1} x_\tau 1_{\{a_\tau = k\}} w_\tau$

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Large when $N_t^{(k)}$ small or x_t not aligned with historical data

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Makes $A_t^{(k)}$ invertible always, and it turns out a bound just like Chebyshev's applies (with more details and a much more complicated proof, which we won't get into)

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For $t = 0 \rightarrow T - 1$

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Can prove $\tilde{O}(\sqrt{T})$ regret bound

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Both cases allow a version of linUCB by extension of the same ideas: fit coefficients via least squares and use Chebyshev-like uncertainty quantification to get UCB

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ii. Good for large K, but step 2's argmax may be hard

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But in principle, there is no "free lunch", i.e., the hardness of the problem now transfers over to choosing a good model (a bad model will lead to bad performance)

Today

- Feedback from last lecture
- Recap
- UCB-VI for linear MDPs
- Recall: Contextual Bandits
- LinUCB

Summary:

- Modeling in MDPs and bandits with large state/action spaces is critical
- When model is linear (in feature space), can still rigorously quantify uncertainty and balance exploration/exploitation

Attendance:
bit.ly/3RcTC9T

Feedback:
bit.ly/3RHtlxy