UCB-VI

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CS/Stat 184(0): Introduction to Reinforcement Learning

Fall 2024

Today

- Feedback from last lecture
- Recap
- Warm-up: ExploreThenExploit for deterministic MDPs
- Why we don’t want to treat MDPs as big bandits
- UCB-VI for tabular MDPs
- UCB-VI for linear MDPs

Feedback from feedback forms

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1. Thank you to everyone who filled out the forms!

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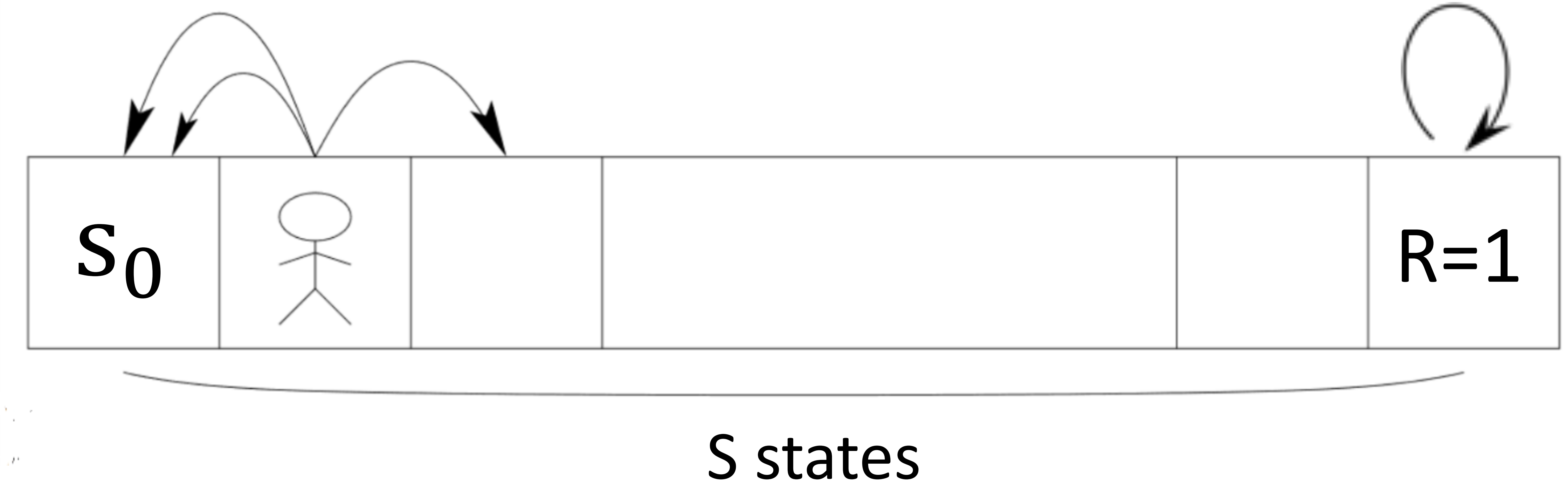
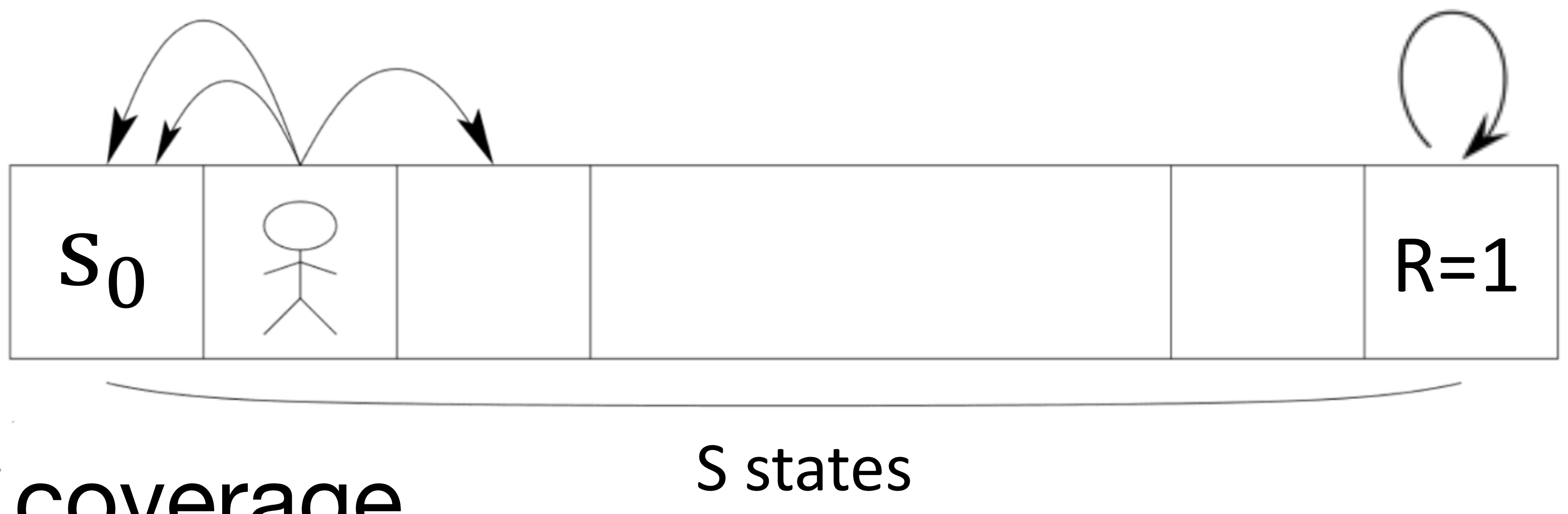
"Lack of Exploration" leads to Optimization and Statistical Challenges



Thrun '92

- Suppose $H \approx \text{poly}(|S|)$ & $\mu(s_0) = 1$ (i.e. we start at s_0).
- A randomly initialized policy π^0 has prob. $O(1/3^{|S|})$ of hitting the goal state in a trajectory.
- Thus a sample-based approach, with $\mu(s_0) = 1,$ require $O(3^{|S|})$ trajectories.
    - Holds for (sample based) Fitted DP
    - Holds for (sample based) PG/TRPO/NPG/PPO
- Basically, for these approaches, there is no hope of learning the optimal policy if $\mu(s_0) = 1.$

Let's examine the role of μ



Thrun '92

- Suppose that somehow the distribution μ had better coverage.
  - e.g, if μ was uniform overall states in our toy problem, then all approaches we covered would work (with mild assumptions)
  - Theory: TRPO/NPG/PPO have better guarantees than fitted DP methods (assuming some "coverage")
- Strategies without coverage:
  - If we have a simulator, sometimes we can design μ to have better coverage.
    - this is helpful for robustness as well.
  - Imitation learning (next time).
    - An expert gives us samples from a "good" μ.
  - Explicit exploration:
    - UCB-VI: we'll merge two good ideas!
    - Encourage exploration in PG methods.
  - Try with reward shaping

Recall: Value Iteration (VI)

VI = DP is a backwards in time approach for computing the optimal policy:

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Recall: Upper Confidence Bound (UCB)

For $t = 0, \ldots, T-1$:

Choose the arm with the highest upper confidence bound, i.e.,

$$a_t = \arg\max_{k \in \{1,\ldots,K\}} \hat{\mu}_t^{(k)} + \sqrt{\ln(2TK/\delta)/2N_t^{(k)}}$$

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High-level summary: estimate action quality, add exploration bonus, then argmax

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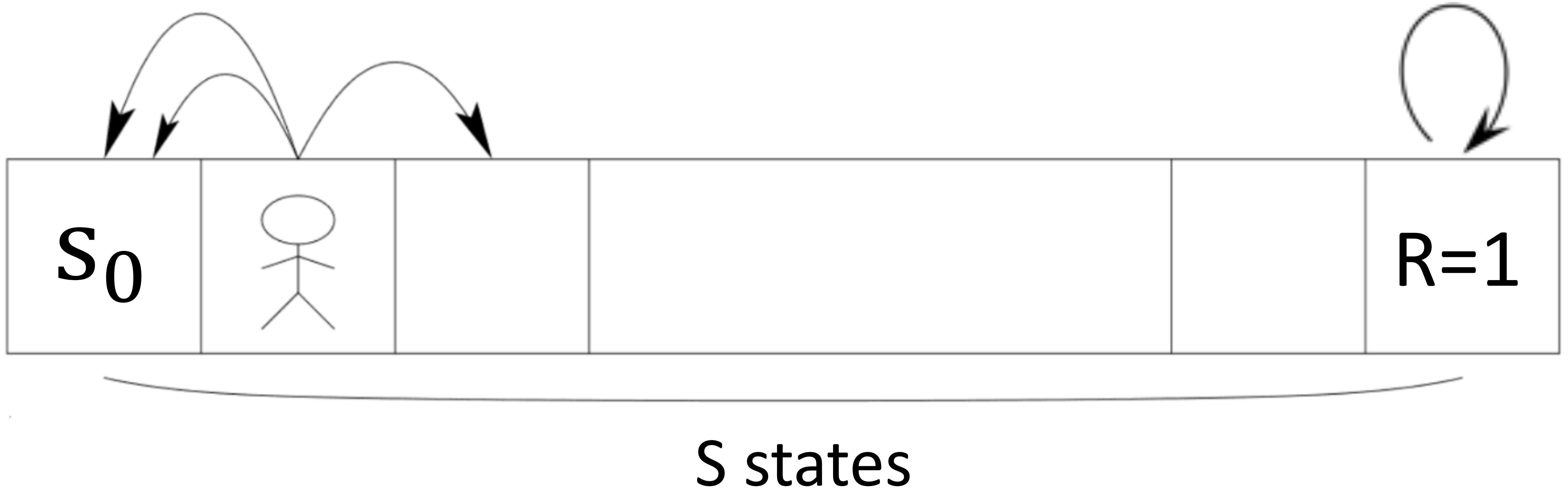
How we do find $\pi^\star$ in an unknown MDP?

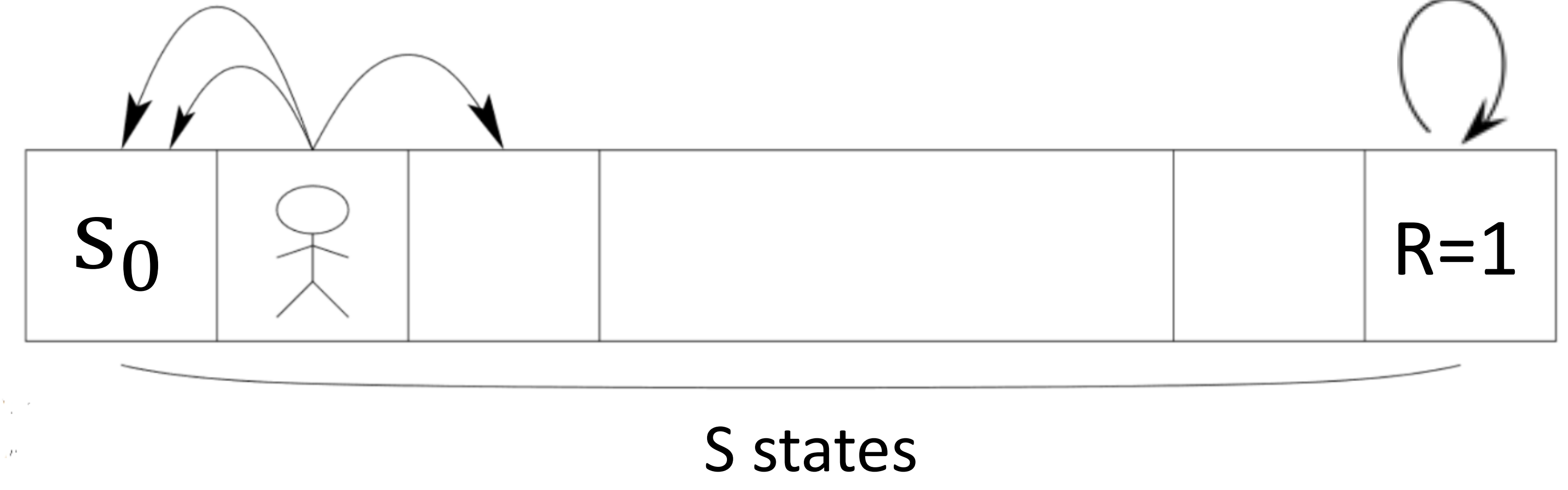


Thrun '92

- Episodic setting with an unknown MDP:
  - suppose we start at $s_0 \sim \mu$.
  - We act for H steps.
  - Then repeat.
- How do we find $\pi^\star$?
- How do we get low regret?

- Let's start with the setting where the MDP is deterministic.
  - So both $r(s, a)$ and $P(\cdot \mid s, a)$ are deterministic.

Algorithm: ExploreThenExploit (for deterministic MDPs)



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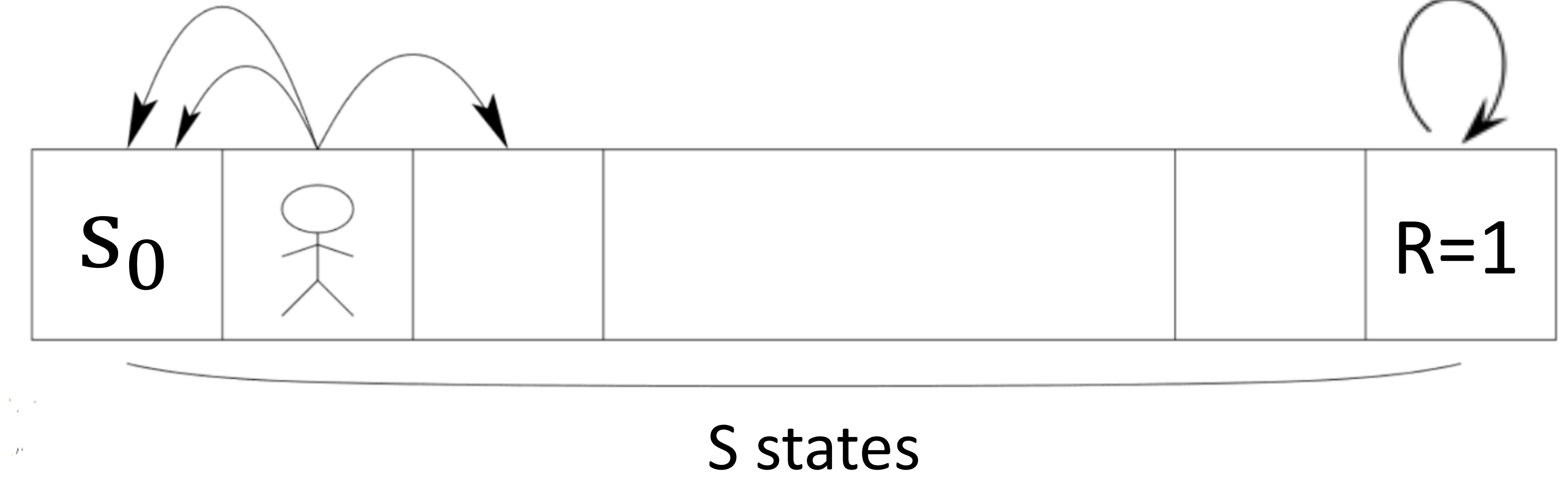

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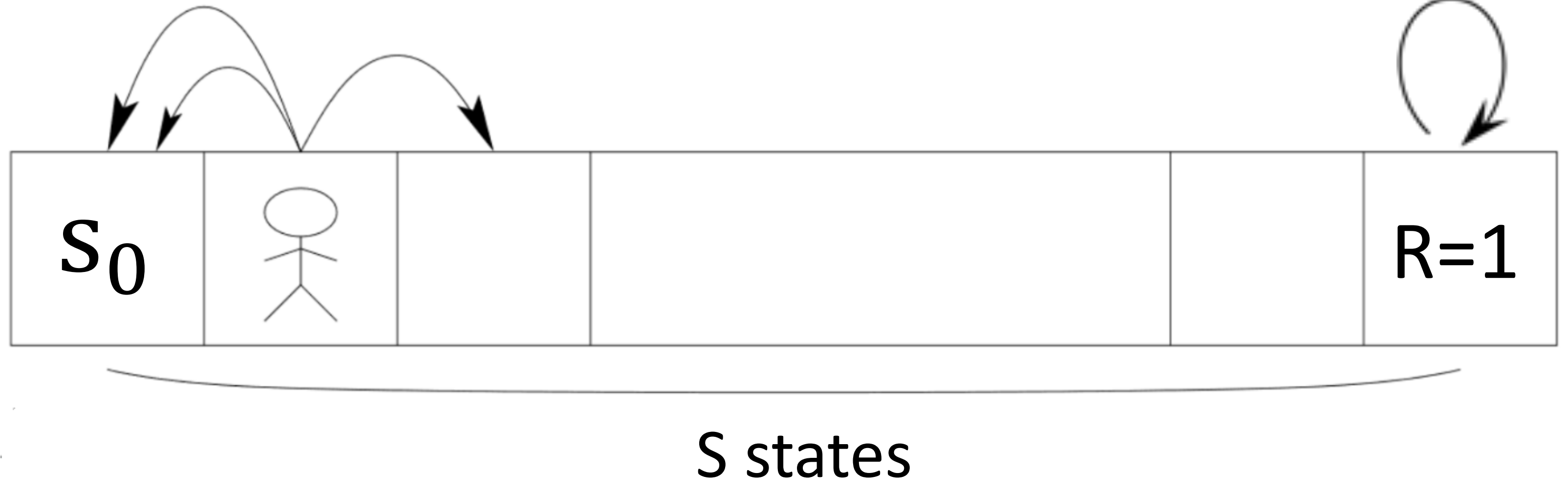

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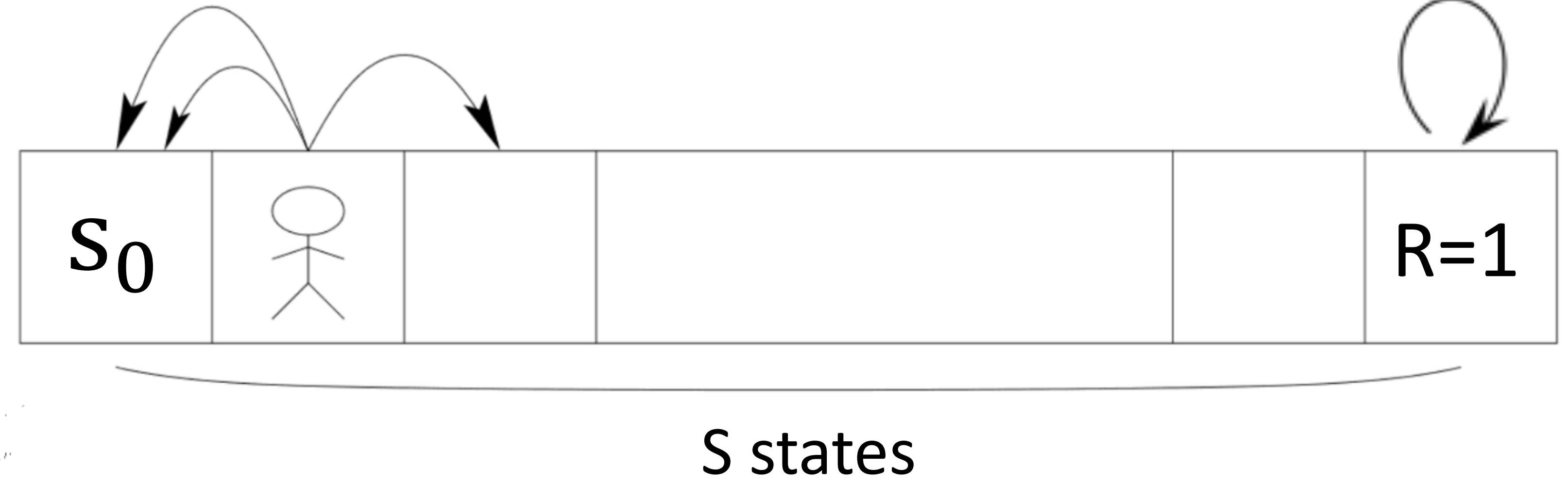

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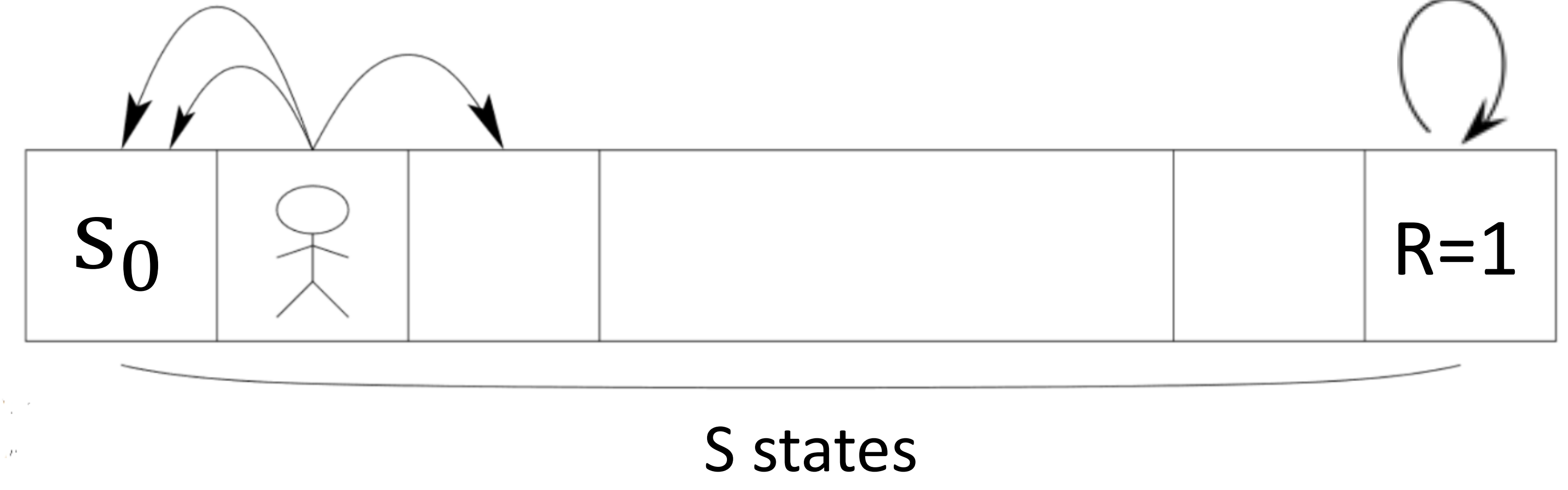

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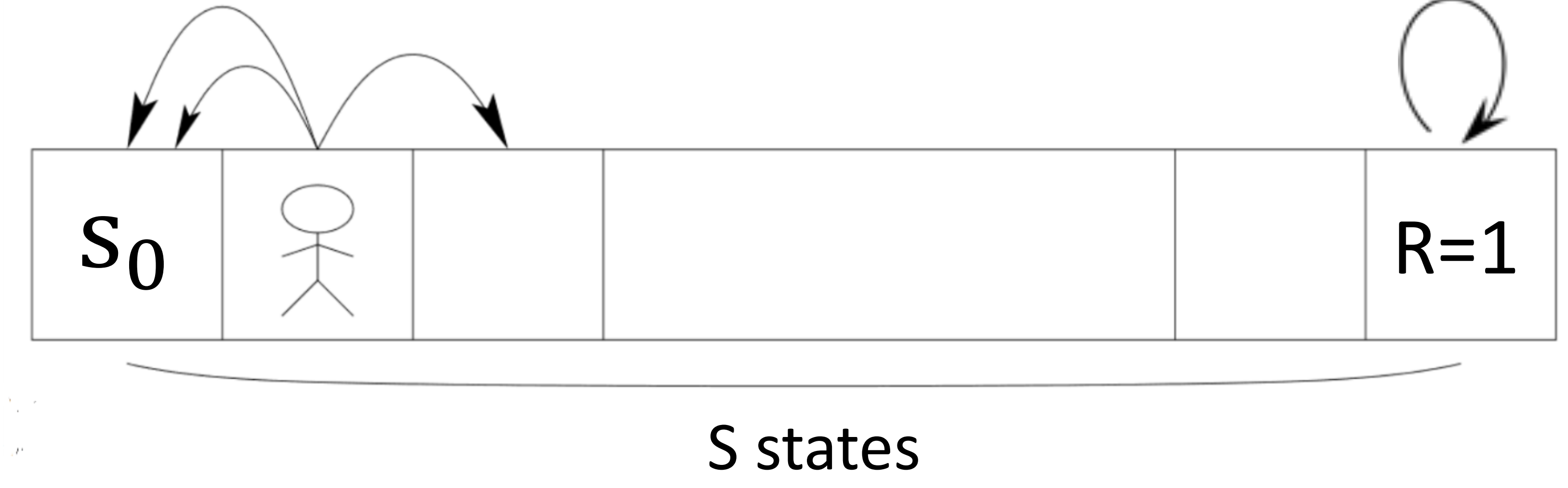

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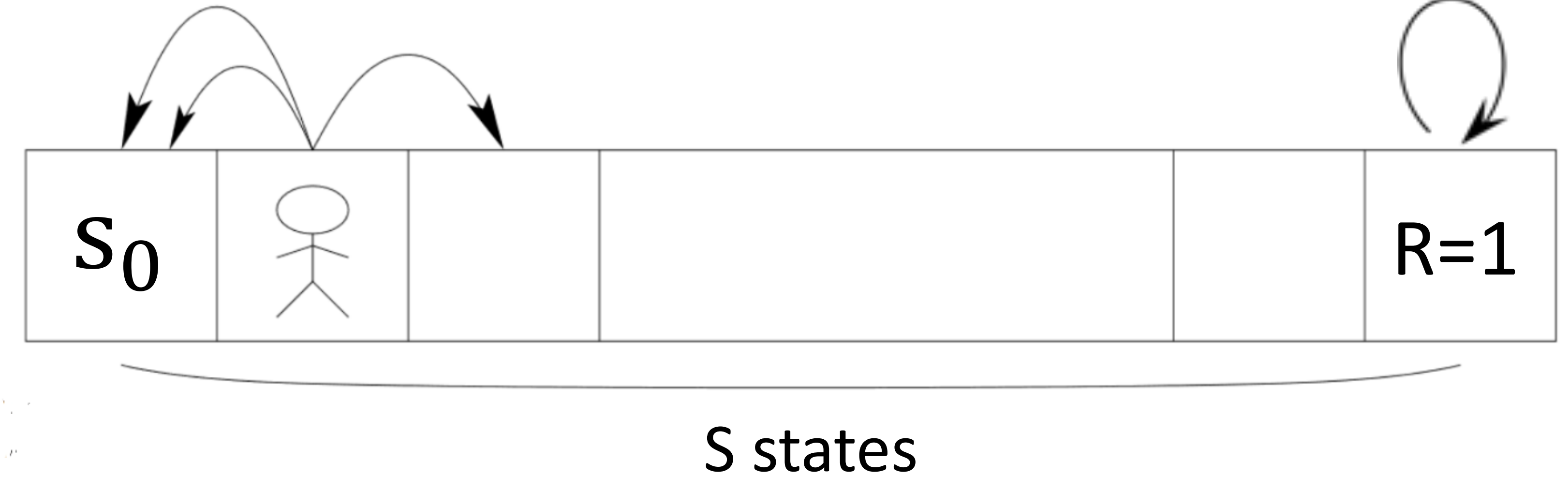

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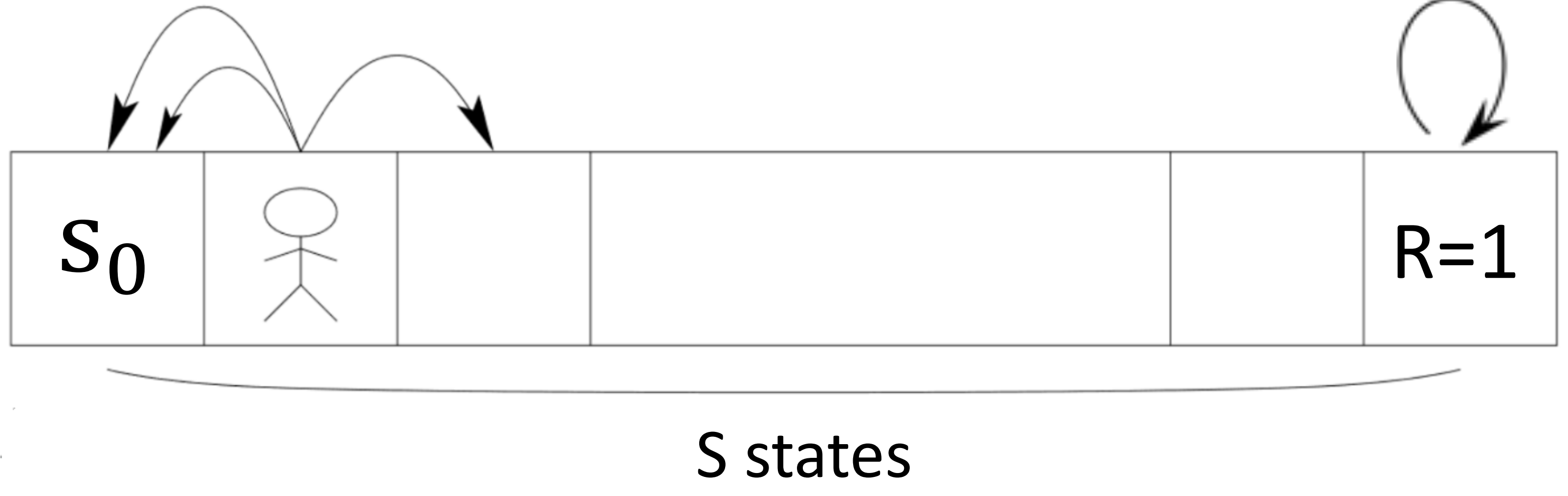

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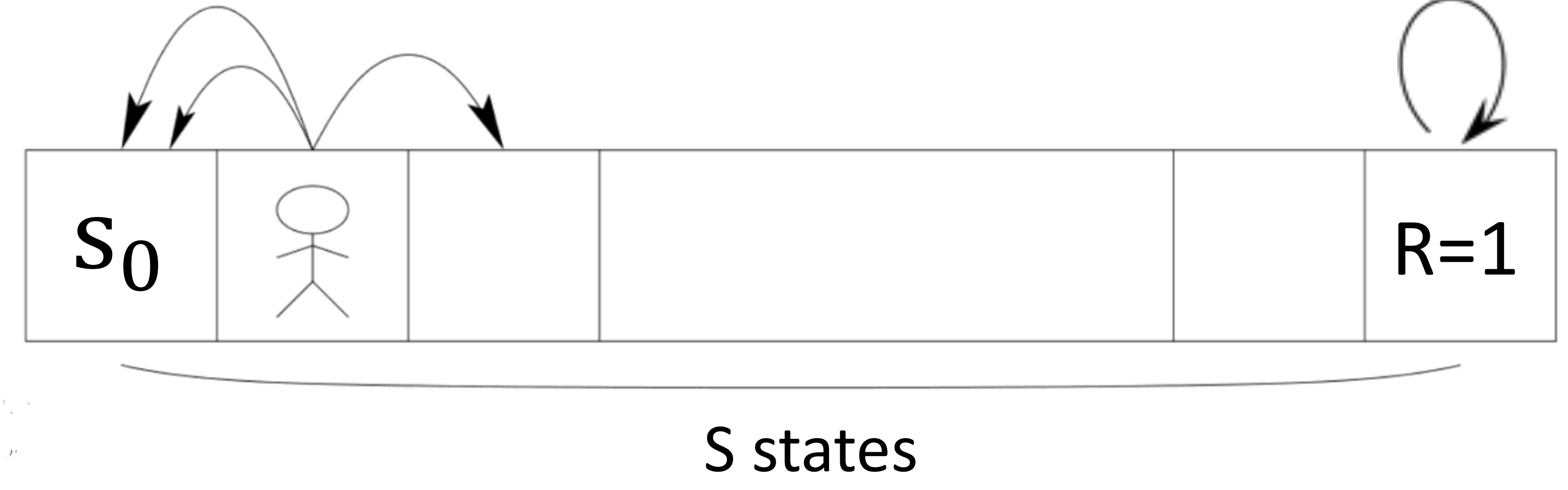

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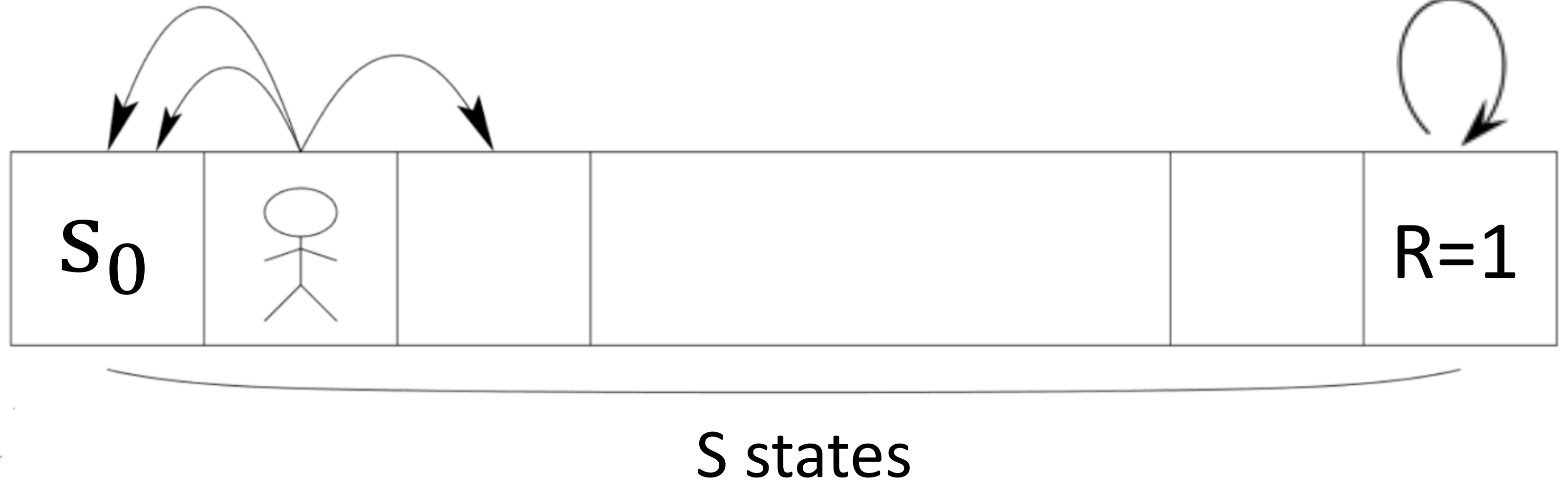

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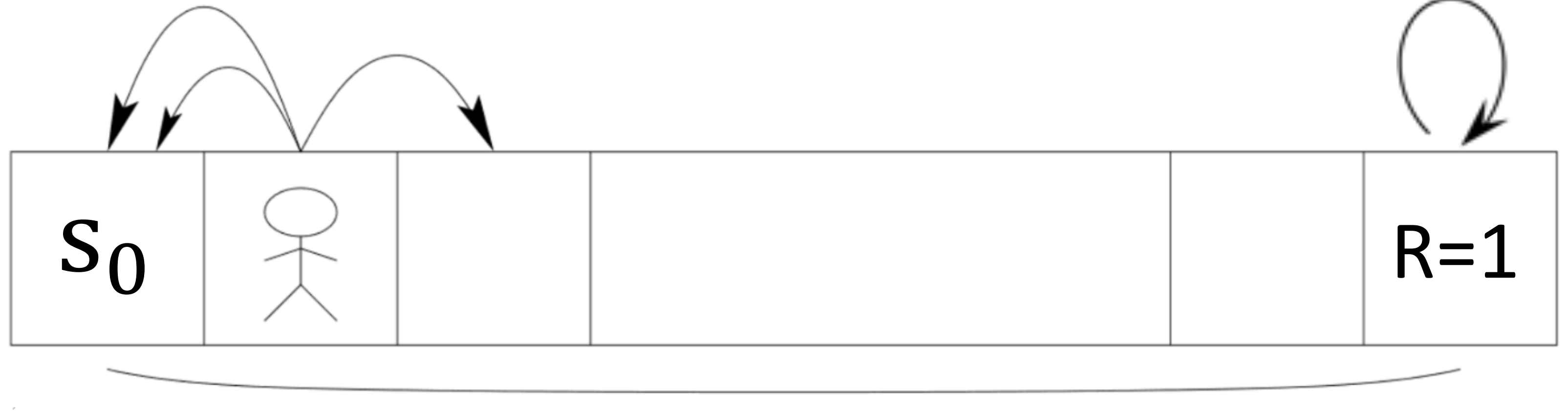

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- Let $\pi_K^\star$ and $V_K^\star$ be the optimal policy and value in M_K.
- Theorem: Assume $H \geq |S|$.
  If K does not contain all state-action pairs, then $V_K^\star > 0$ and $\pi_K^\star$ will reach some $(s, a) \notin K$ (in at most $|S|$ steps).

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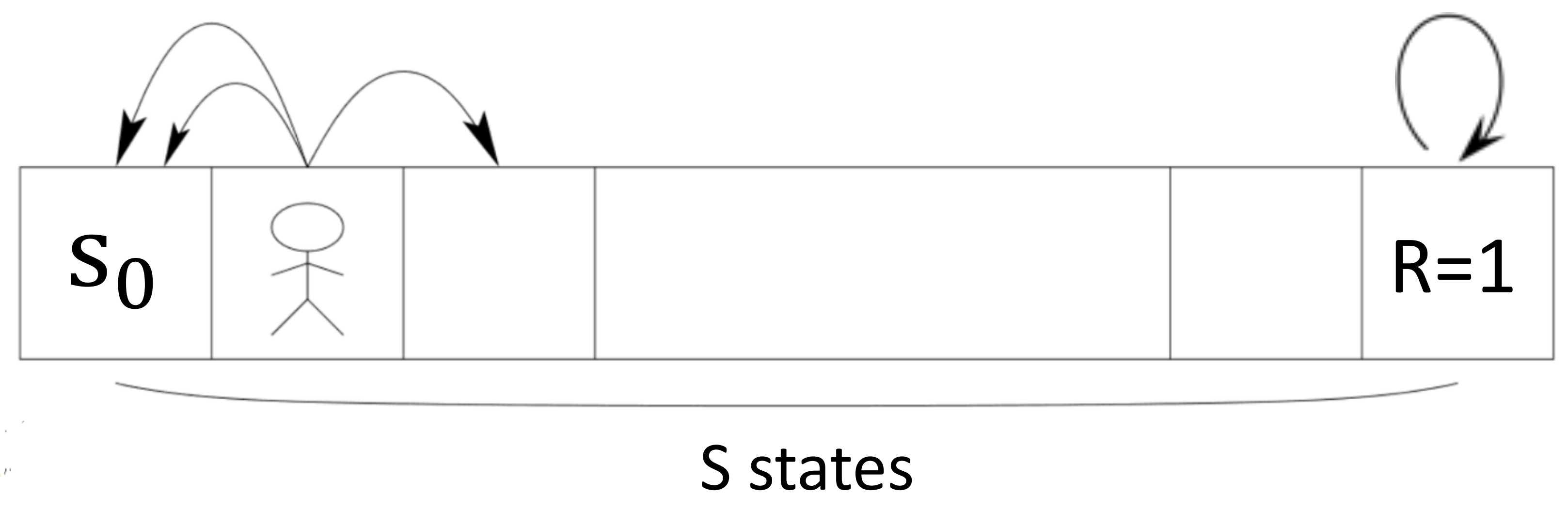

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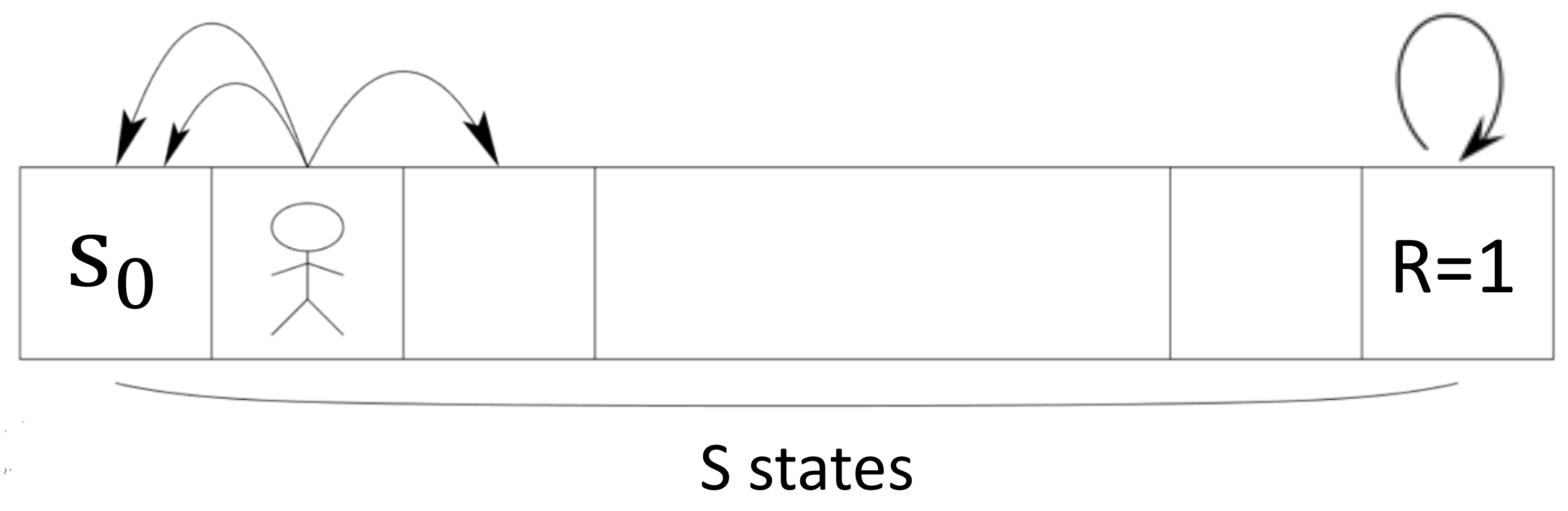

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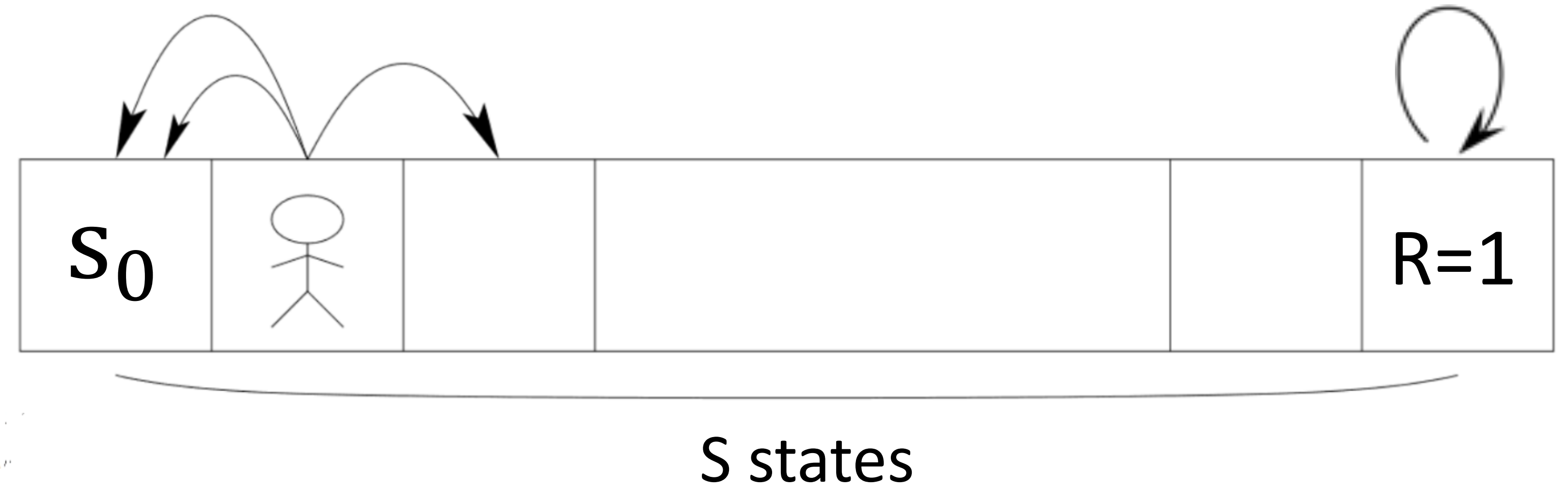

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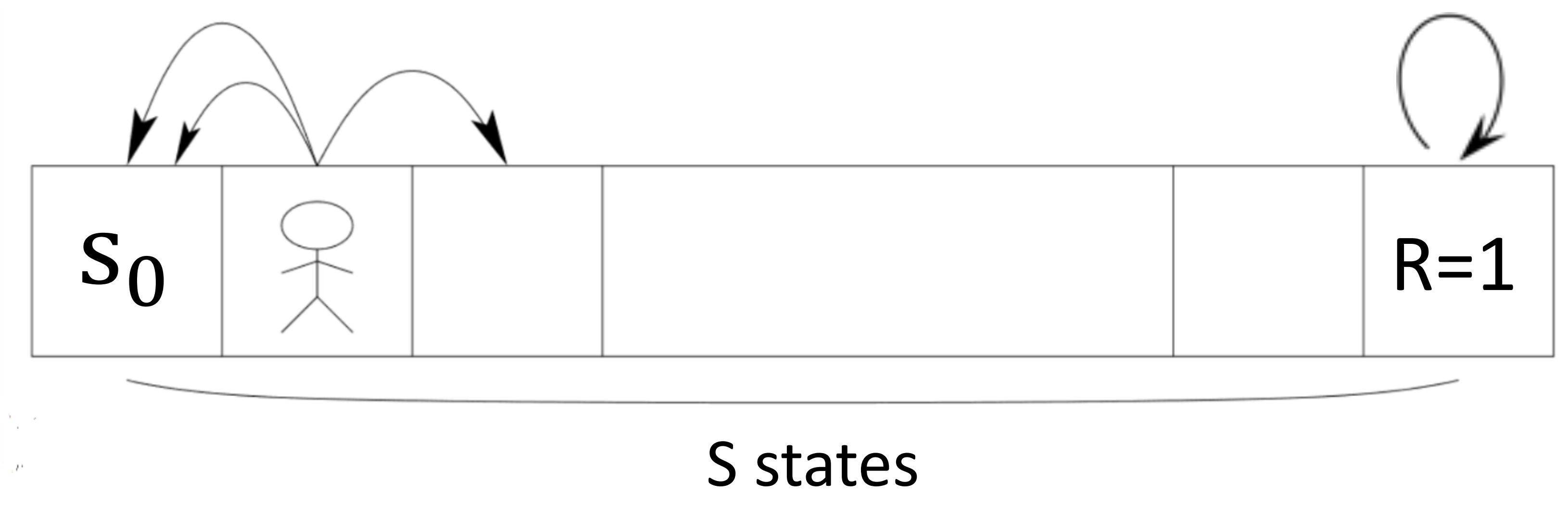

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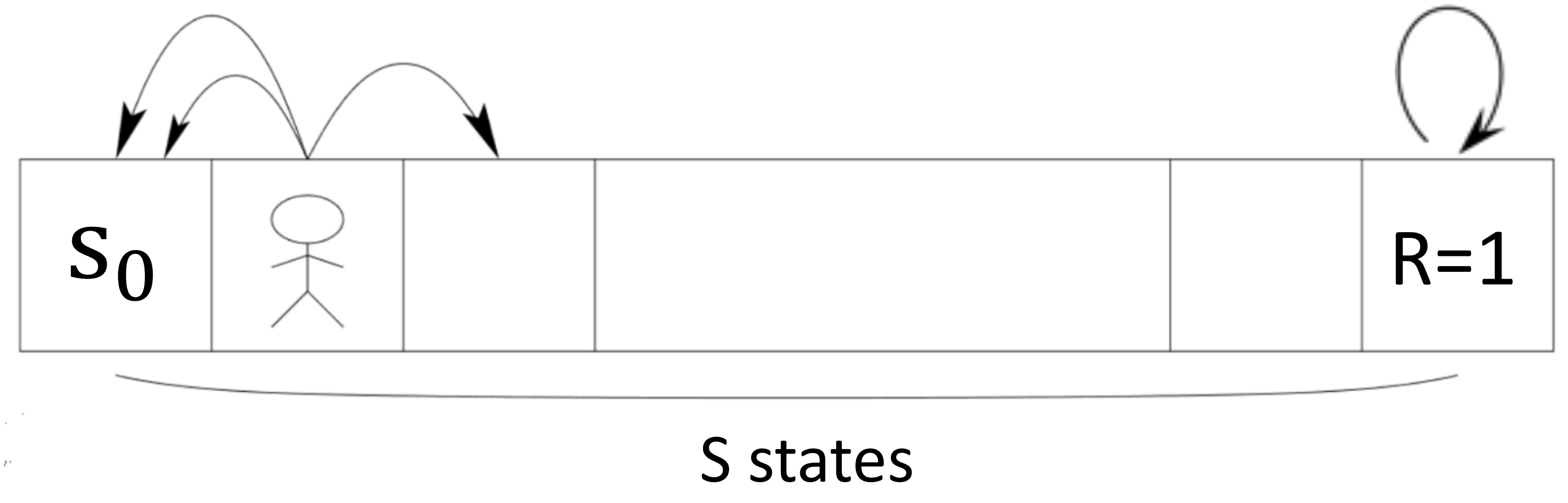

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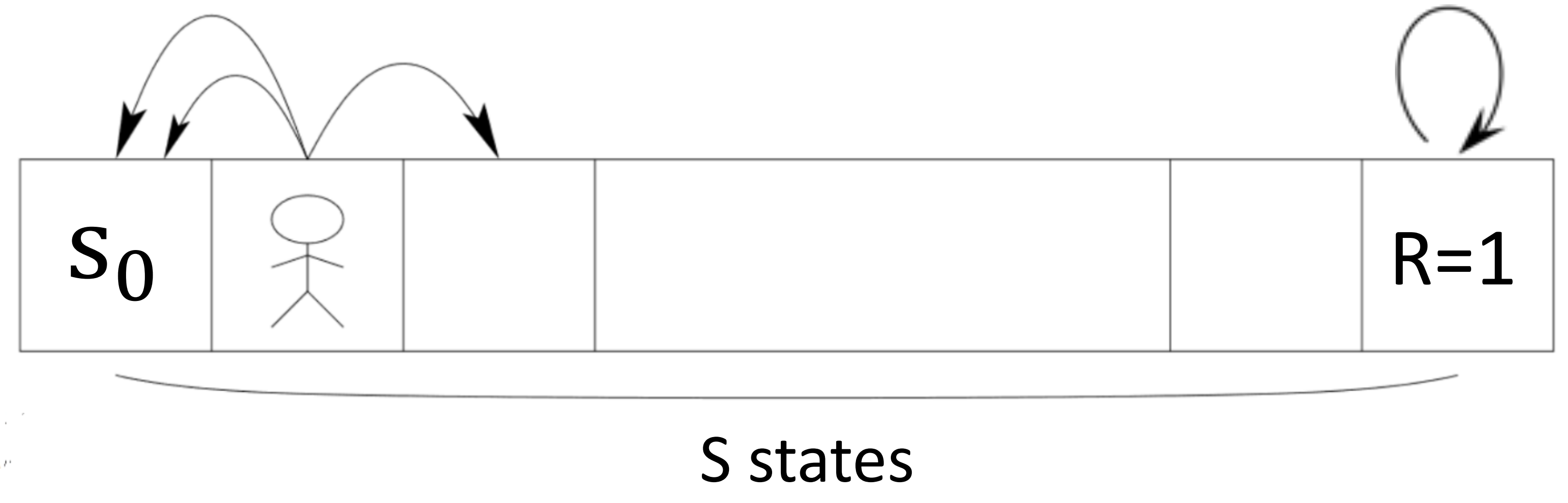

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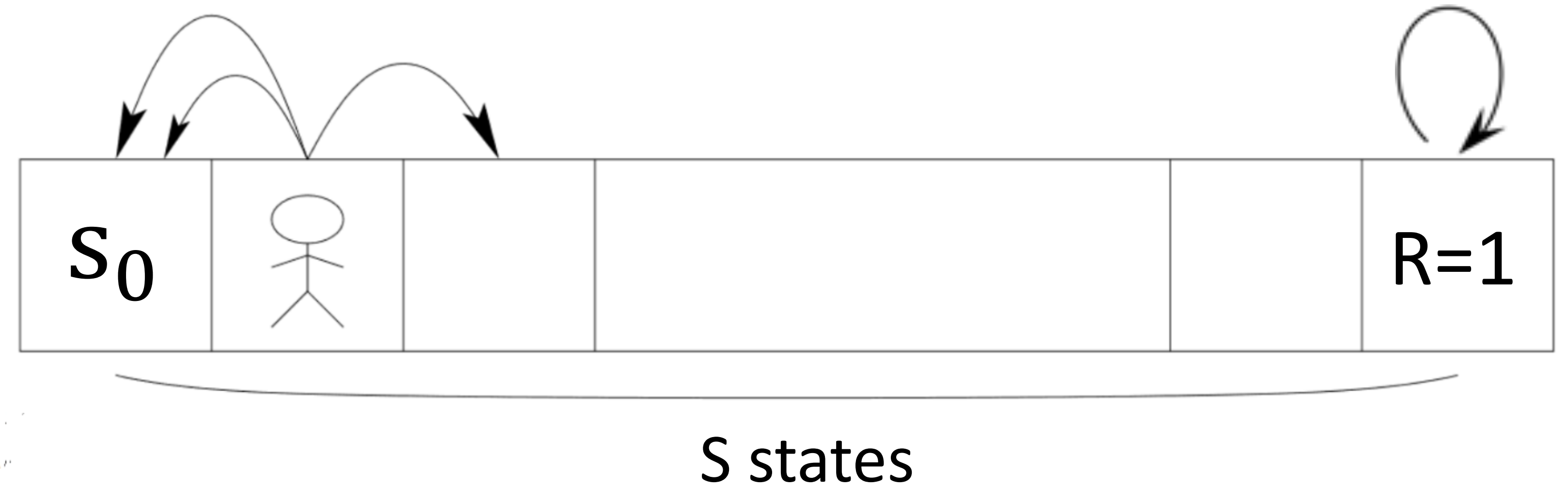

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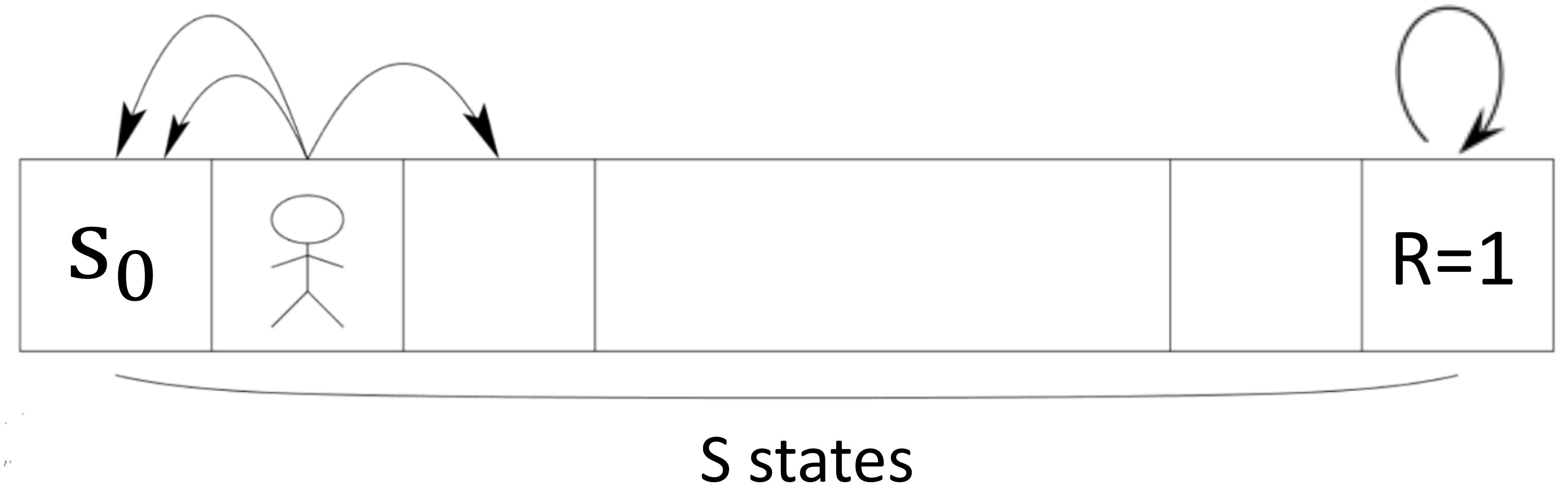

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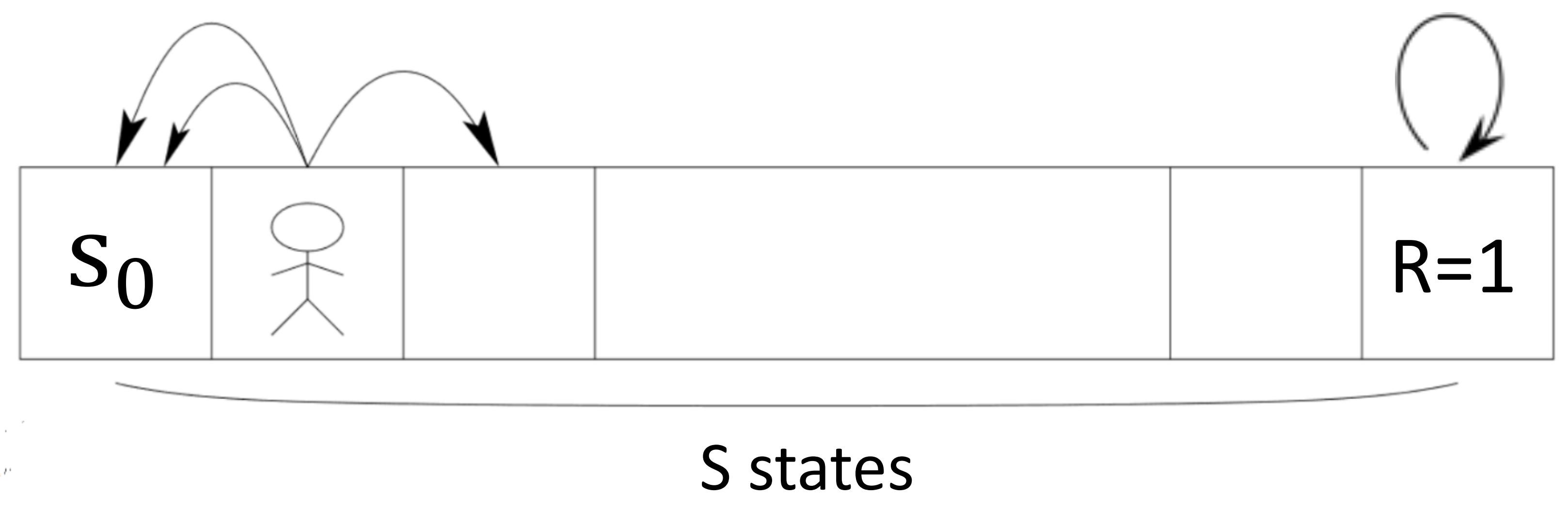

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    - If $V^\star_K > 0$, execute $\pi^\star_K$ and update the known set K
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- Return: the optimal policy in the known MDP.

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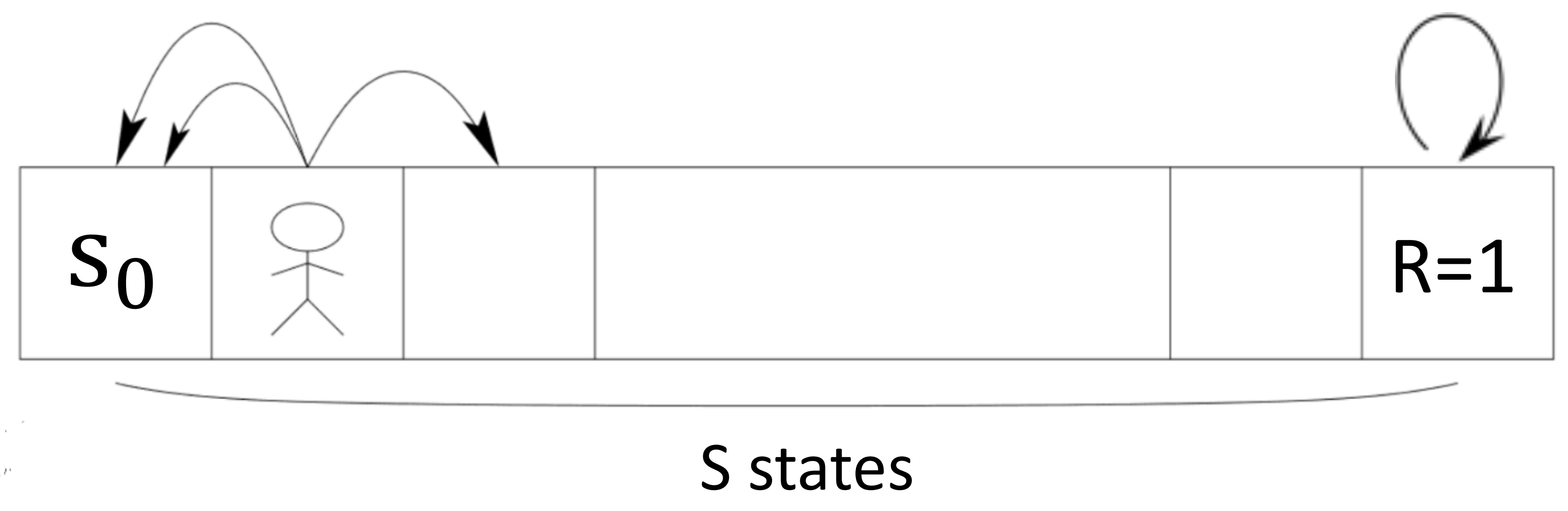

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Theorem: Assuming $H \geq |S|$, this algorithm returns an optimal policy in most $|S| \cdot |A|$ trajectories.

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- How do we modify the algorithm for general H?
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- What is the regret of this algorithm?

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- Really needed determinism; for non-deterministic MDPs, need to think more like bandits…

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- ✓ Warm-up: ExploreThenExploit for deterministic MDPs
- Why we don’t want to treat MDPs as big bandits
- UCB-VI for tabular MDPs
- UCB-VI for linear MDPs

Exploration in MDP: make it a bandit and do UCB?

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So treating each policy as an "arm" and running UCB gives us regret $\tilde{O}(\sqrt{|A|^{|S|H}N})$

This seems bad, so are MDPs just super hard or can we do better?

An example of MDP as bandit

$$S = \{a, b\}, \quad A = \{1,2\}, \quad H = 2$$

All state transitions happen with probability 1/2 for all actions

Reward function: $$\begin{aligned} r(a,1) = r(b,1) = 0 \\ r(a,2) = r(b,2) = 1 \end{aligned}$$

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$|A|^{|S|H} = 2^4 = 16$

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Collect a new trajectory by executing π^n in the true system $\{P_h\}_{h=0}^{H-1}$ starting from s_0

Model Estimation

Let us consider the **very beginning** of episode n:

$$\mathcal{D}_h^n = \{s_h^i, a_h^i, s_{h+1}^i\}_{i=1}^{n-1}, \forall h$$

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Estimate model $\hat{P}_h^n(s' \mid s,a), \forall s, a, s', h$:

$$\hat{P}_h^n(s' \mid s,a) = \frac{N_h^n(s,a,s')}{N_h^n(s,a)}$$

Reward Bonus Design and Value Iteration

Recall: $\mathscr{D}_h^n = \{s_h^i, a_h^i, s_{h+1}^i\}_{i=1}^{n-1}, \forall h, \quad N_h^n(s,a) = \sum_{i=1}^{n-1} \mathbf{1}\{(s_h^i, a_h^i) = (s,a)\}, \forall s, a, h,$

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$$\left\| \hat{V}_h^n \right\|_\infty \leq H, \ \forall h, n$$

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$b_h^n(s,a)$ specifically chosen so that $V_h^\star(s) \leq \hat{V}_h^n(s)$ with high probability

UCBVI: Put All Together

For $n = 1 \to N$:

1. Set $N_h^n(s,a) = \sum_{i=1}^{n-1} \mathbf{1}\{(s_h^i, a_h^i) = (s,a)\}, \forall s, a, h$

2. Set $N_h^n(s,a,s') = \sum_{i=1}^{n-1} \mathbf{1}\{(s_h^i, a_h^i, s_{h+1}^i) = (s,a,s')\}, \forall s, a, a', h$

3. Estimate $\hat{P}^n$: $\hat{P}_h^n(s' \mid s,a) = \dfrac{N_h^n(s,a,s')}{N_h^n(s,a)}, \forall s, a, s', h$

4. Plan: π^n = VI $\left(\{\hat{P}_h^n, r_h + b_h^n\}_h \right)$, with $b_h^n(s,a) = cH\sqrt{\dfrac{\log(|S||A|HN/\delta)}{N_h^n(s,a)}}$

5. Execute π^n : $\{s_0^n, a_0^n, r_0^n, \ldots, s_{H-1}^n, a_{H-1}^n, r_{H-1}^n, s_H^n\}$

High-level Idea: Exploration Exploitation Tradeoff

Upper bound per-episode regret: $V_0^\star(s_0) - V_0^{\pi^n}(s_0) \leq \hat{V}_0^n(s_0) - V_0^{\pi^n}(s_0)$ by construction of b_h^n

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$$\mathbb{E}\left[\text{Regret}_N\right] := \mathbb{E}\left[\sum_{n=1}^{N}\left(V^\star - V^{\pi^n}\right)\right] \leq \widetilde{O}\left(H^2\sqrt{|S||A|N}\right)$$

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Linear MDP Definition

Finite horizon time-dependent episodic MDP $\mathscr{M} = \{S, A, H, \{r\}_h, \{P\}_h, s_0\}$

S & A could be large or even continuous, hence poly$(|S|, |A|)$ is not acceptable

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$$P_h(s'|s,a) = \mu_h^\star(s') \cdot \phi(s,a), \quad \mu_h^\star : S \mapsto \mathbb{R}^d, \quad \phi : S \times A \mapsto \mathbb{R}^d$$

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Feature map ϕ is known to the learner!

(We assume reward is known, i.e., $\theta^\star$ is known)

Planning in Linear MDP: Value Iteration

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Is linear with respect to ϕ as well, for any π, h

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How to choose $b_h^n(s,a)$?

Chebyshev-like approach, similar to in linUCB (will cover next lecture):

$$b_h^n(s,a) = \beta\sqrt{\phi(s,a)^\top (A_h^n)^{-1}\phi(s,a)}, \quad \beta = \widetilde{O}(dH)$$

linUCB-VI: Put All Together

For $n = 1 \to N$:

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5. Execute $\pi^n : \{s_0^n, a_0^n, r_0^n, \ldots, s_{H-1}^n, a_{H-1}^n, r_{H-1}^n, s_H^n\}$

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No S, A dependence!

Today

- Feedback from last lecture
- Recap
- Warm-up: ExploreThenExploit for deterministic MDPs
- Why we don't want to treat MDPs as big bandits
- UCB-VI for tabular MDPs
- UCB-VI for linear MDPs

Summary:

UCBVI algorithm applies UCB idea to MDPs to achieve exploration/exploitation trade-off

Attendance:

bit.ly/3RcTC9T

Feedback:

bit.ly/3RHtlxy