Trust Region Policy Optimization & The Natural Policy Gradient

Lucas Janson

CS/Stat 184(0): Introduction to Reinforcement Learning
Fall 2024

Today

- Feedback from last lecture
- Recap
- The Performance Difference Lemma
- Trust Region Policy Optimization (TRPO)
- The Natural Policy Gradient (NPG)

Feedback from feedback forms

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2. Discuss projects!

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Optimization Objective

- Consider a parameterized class of policies:
  $$\{\pi_\theta(a \mid s) \mid \theta \in \mathbb{R}^d\}$$
  (why do we make it stochastic?)

- Objective $\max_\theta J(\theta)$, where
  $$J(\theta) := \mathbb{E}_{s_0 \sim \mu}\left[V^{\pi_\theta}(s_0)\right] = \mathbb{E}_{\tau \sim \rho_{\pi_\theta}}\left[\sum_{h=0}^{H-1} r(s_h, a_h)\right]$$

- Policy Gradient Descent:
  $$\theta^{k+1} = \theta^k + \eta \nabla J(\theta^k)$$

REINFORCE: A Policy Gradient Algorithm

- Let $\rho_\theta(\tau)$ be the probability of a trajectory $\tau = \{s_0, a_0, s_1, a_1, \ldots, s_{H-1}, a_{H-1}\}$, i.e.
$$\rho_\theta(\tau) = \mu(s_0)\pi_\theta(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\ldots P(s_{H-1} \mid s_{H-2}, a_{H-2})\pi_\theta(a_{H-1} \mid s_{H-1})$$

- Let $R(\tau)$ be the cumulative reward on trajectory τ, i.e. $R(\tau) := \sum_{h=0}^{H-1} r(s_h, a_h)$

- Our objective function is:
$$J(\theta) \;= E_{\tau\sim\rho_\theta}\Big[R(\tau)\Big]$$

- From the likelihood ratio method, we have:
$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau\sim\rho_\theta}\left[\nabla_\theta \ln \rho_\theta(\tau)\; R(\tau)\right]$$

- The REINFORCE Policy Gradient expression:
$$\nabla_\theta \ln \rho_\theta(\tau)\; R(\tau) = \left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)$$

Obtaining an Unbiased Gradient Estimate at θ

$$\nabla_\theta J(\theta) := \mathbb{E}_{\tau \sim \rho_\theta}\left[\left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)\right]$$

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We have: $\mathbb{E}[g(\theta, \tau)] = \nabla_\theta J(\theta)$

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    2. Update: $\theta^{k+1} = \theta^k + \eta^k g(\theta^k, \tau)$

Other PG formulas
(that are lower variance for sampling)

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta}\left[\left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)\right]$$ (REINFORCE)

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Intuition: Changing the action distribution at h only affects rewards later on…

HW: You will show these simplified version are also valid PG expressions

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For any function only of the state, $b_h : S \to \mathbb{R}$, we have:

This is (basically) the method of control variates.

- For the proof, it was helpful to note:
  $\mathbb{E}_{x \sim P_\theta}\left[\nabla_\theta \log P_\theta(x)\ c\right] = 0$

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The Advantage Function (finite horizon)

$$V_h^\pi(s) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \,\middle|\, s_h = s\right] \qquad Q_h^\pi(s,a) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \,\middle|\, (s_h, a_h) = (s,a)\right]$$

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- For the discounted case, $A^\pi(s,a) = Q^\pi(s,a) - V^\pi(s)$

The Advantage-based PG:

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta(\tau)} \left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) \left(Q_h^{\pi_\theta}(s_h, a_h) - b_h(s_h) \right) \right]$$

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- In practice, the most common approach is to use $b_h(s)$ that's an estimate of $V_h^\pi(s)$.

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Note that regardless of our choice of $\widetilde{b}$, we still get unbiased gradient estimates.

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Recall: Fitted Policy Iteration

- Initialization: choose a policy $\pi^0 : S \mapsto A$ and a sample size N
- For $k = 0,1,\ldots$
  1. **Fitted Policy Evaluation**: Using N sampled trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi^k}$, obtain approximation $\hat{Q}^{\pi^k} \approx Q^{\pi^k}$
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- Helps us think about error analysis, instabilities of fitted PI, and sub-optimality.
- Helps to understand algorithm design (TRPO, NPG, PPO)
- This also motivates the use of “local” methods (e.g. policy gradient descent)

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- In particular, $\hat{A}^{\pi^k}$ should be close to A^{π^k} where π^k visits often…

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- This means we expect $\mathbb{E}_{\tau \sim \rho_{\pi^k,s}} \left[\sum_{h=0}^{H-1} \hat{A}^{\pi^k}(s_h, a_h, h) \right] \approx \mathbb{E}_{\tau \sim \rho_{\pi^k,s}} \left[\sum_{h=0}^{H-1} A^{\pi^k}(s_h, a_h, h) \right]$
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- One way to ensure this: keep $\pi^{k+1} \approx \pi^k$

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   try to approximately solve:
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- How should we define "close", i.e., what is our "trust" region?

KL-divergence: measures the distance between two distributions

Given two distributions P & Q, where $P \in \Delta(X), Q \in \Delta(X)$,
KL Divergence is defined as:

$$KL(P \mid Q) = \mathbb{E}_{x \sim P}\left[\ln \frac{P(x)}{Q(x)}\right]$$

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Fact:

$KL(P \mid Q) \geq 0$, and is 0 if and only if $P = Q$

Trust Region Policy Optimization (TRPO)

1. Initialize θ^0 , δ
2. For $k = 0, \ldots, K$:
   try to approximately solve:

$$\theta^{k+1} = \arg\max_{\theta} \mathbb{E}_{s_0,\ldots,s_{H-1}\sim\rho_{\pi_{\theta^k}}} \left[\sum_{h=0}^{H-1} \mathbb{E}_{a_h\sim\pi_\theta(\cdot|s_h)} \left[A^{\pi_{\theta^k}}(s_h, a_h, h)\right] \right]$$

$$\text{s.t. } KL\left(\rho_{\pi_{\theta^k}} | \rho_{\pi_\theta}\right) \leq \delta$$

3. Return π_{θ^K}

- We want to maximize local advantage against π_{θ^k},
  but we want the new policy to be close to π_{θ^k} (in the KL sense)
- How do we implement this with sampled trajectories?)

How do we implement TRPO with samples?

1. Initialize parameter θ^0, sample size M, and tolerance δ
2. For $k = 0, \ldots, K$:
   1. [Advantage-Evaluation Subroutine]
      Using M sampled trajectories $\tau_1, \ldots \tau_M \sim \rho_{\pi_{\theta^k}}$, obtain approximation $\hat{A}^{\pi_{\theta^k}} \approx A^{\pi_{\theta^k}}$
   2. Solve the following optimization problem to obtain θ^{k+1}:

$$\max_{\theta} \sum_{m=1}^{M} \sum_{h=0}^{H-1} \mathbb{E}_{a \sim \pi_\theta(\cdot | s_h^m)} \left[\hat{A}^{\pi_{\theta^k}}(s_h^m, a, h) \right]$$

$$\text{s.t. } \sum_{m=1}^{M} \sum_{h=0}^{H-1} \ln \frac{\pi_{\theta^k}(a_h^m \mid s_h^m)}{\pi_\theta(a_h^m \mid s_h^m)} \leq \delta$$

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Approximate expectation by importance sampling:

$$\mathbb{E}_{a \sim \pi_\theta(\cdot|s_h^m)} \left[\hat{A}^{\pi_{\theta^k}}(s_h^m, a, h) \right]$$

$$= \mathbb{E}_{a \sim \pi_{\theta^k}(\cdot|s_h^m)} \left[\frac{\pi_\theta(a | s_h^m)}{\pi_{\theta^k}(a | s_h^m)} \hat{A}^{\pi_{\theta^k}}(s_h^m, a, h) \right]$$

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TRPO is locally equivalent to a much simpler algorithm

TRPO at iteration k:

$$\max_{\theta} \mathbb{E}_{s_0,\ldots,s_{H-1}\sim\rho_{\pi_{\theta^k}}} \left[\sum_{h=0}^{H-1} \mathbb{E}_{a_h\sim\pi_\theta(\cdot|s_h)} \left[A^{\pi_{\theta^k}}(s_h, a_h, h) \right] \right]$$

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Intuition: maximize local advantage subject to being incremental (in KL)

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s.t. $KL\left(\rho_{\pi_{\theta^k}} | \rho_{\pi_\theta}\right) \leq \delta$ → second-order Taylor expansion at θ^k

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$$\max_{\theta} \nabla_\theta J(\theta^k)^\top (\theta - \theta^k)$$

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(Where F_{θ^k} is the "Fisher Information Matrix")

Natural Policy Gradient (NPG): A "leading order" equivalent program to TRPO:

1. Initialize θ^0
2. For $k = 0, \ldots, K$:

$$\theta^{k+1} = \arg\max_{\theta} \nabla_\theta J(\theta^k)^\top (\theta - \theta^k)$$

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- Where $\nabla_\theta J(\theta^k)$ is the gradient of $J(\theta)$ evaluated at θ^k, and
- F_θ is (basically) the Fisher information matrix at $\theta \in \mathbb{R}^d$, defined as:

$$F_\theta := \mathbb{E}_{\tau \sim \rho_{\pi_\theta}} \left[\nabla_\theta \ln \rho_\theta(\tau) \left(\nabla_\theta \ln \rho_\theta(\tau) \right)^\top \right] \in \mathbb{R}^{d \times d}$$

$$= \mathbb{E}_{\tau \sim \rho_{\pi_\theta}} \left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) \left(\nabla_\theta \ln \pi_\theta(a_h \mid s_h) \right)^\top \right]$$

NPG has a closed form update!

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Where $\eta = \sqrt{\dfrac{\delta}{\nabla_\theta J(\theta^k)^\top F_{\theta^k}^{-1} \nabla_\theta J(\theta^k)}}$

An Implementation: Sample Based NPG

1. Initialize θ^0
2. For $k = 0, \ldots, K$:
   - Obtain approximation of Policy Gradient: $\hat{g} \approx \nabla_\theta J(\theta^k)$
   - Obtain approximation of Fisher information: $\hat{F} \approx F_{\theta^k}$
   - Natural Gradient Ascent: $\theta^{k+1} = \theta^k + \eta \hat{F}^{-1} \hat{g}$
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(We will implement it in HW4 on Cartpole)

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- The Performance Difference Lemma
- Trust Region Policy Optimization (TRPO)
- The Natural Policy Gradient (NPG)

Summary:

1. Performance Difference Lemma tells us we need to stay local
2. TRPO and NPG ensure we don't move too much each step

Attendance:
bit.ly/3RcTC9T

Feedback:
bit.ly/3RHtlxy

Example of Natural Gradient on 1-d problem: 2 actions, 1 state

$$(\pi_\theta[1], \pi_\theta[2]) := \left(\frac{\exp(\theta)}{1 + \exp(\theta)}, \frac{1}{1 + \exp(\theta)} \right)$$

$$J(\theta) = 100 \cdot \pi_\theta[1] + 1 \cdot \pi_\theta[2]$$

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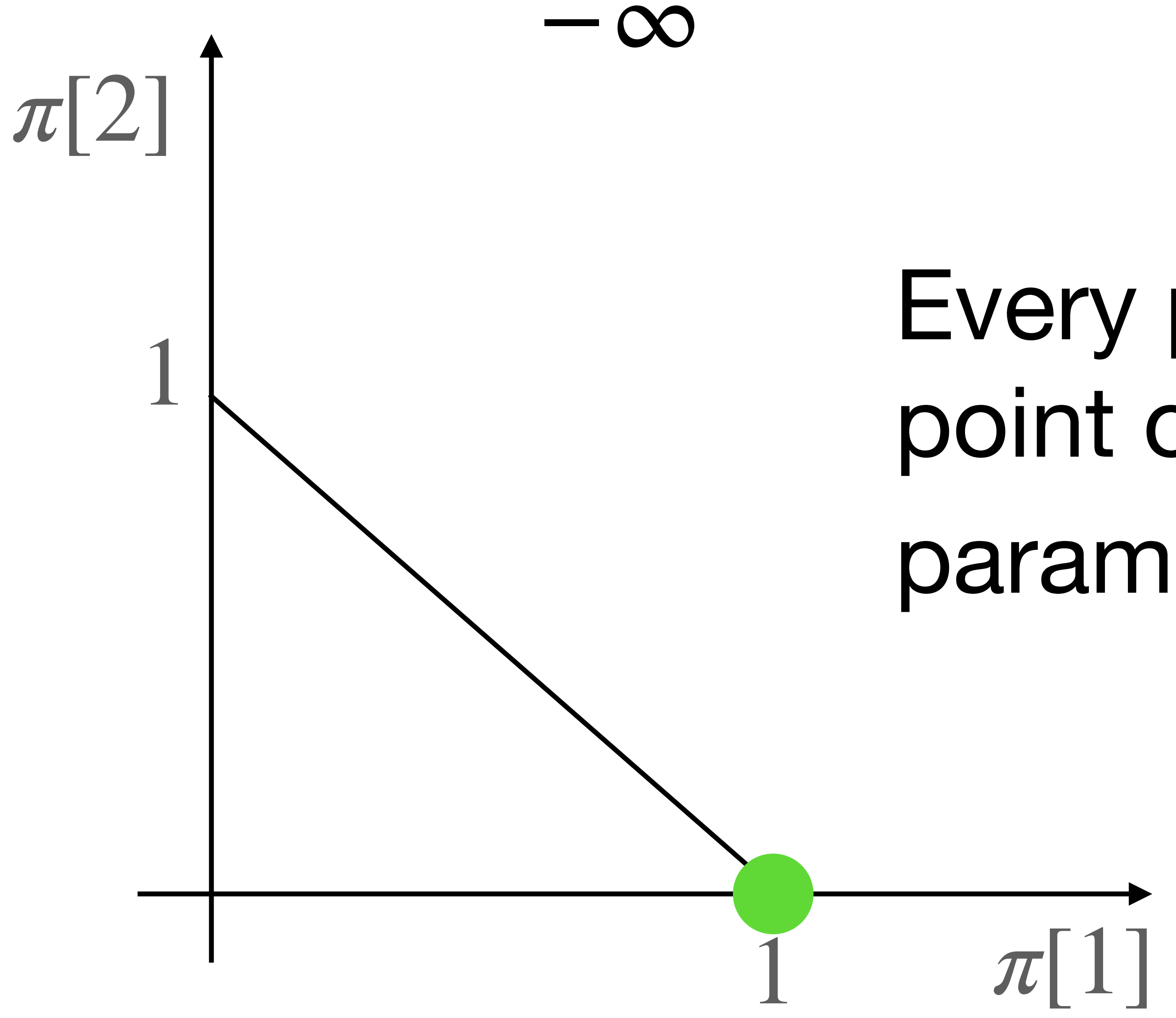

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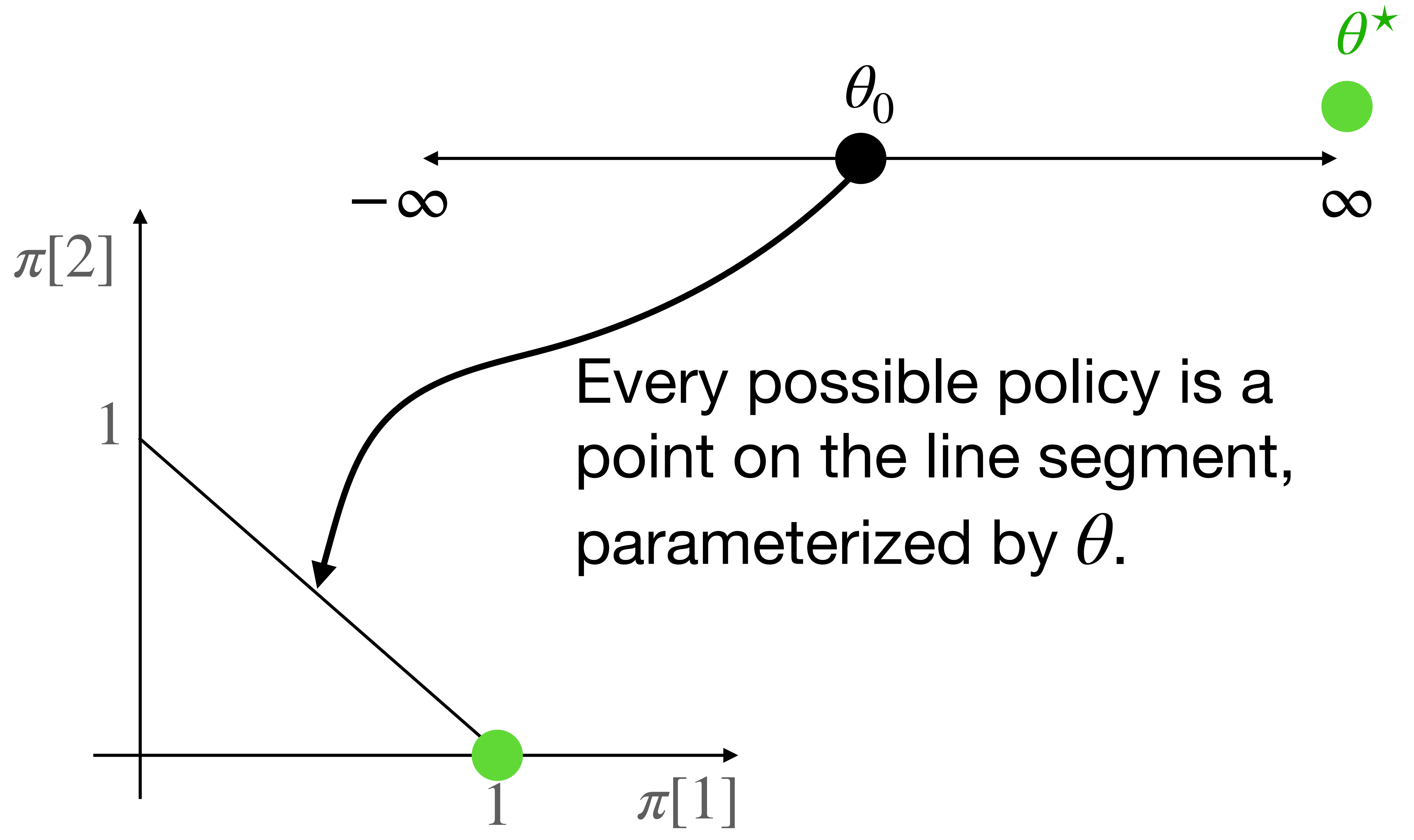

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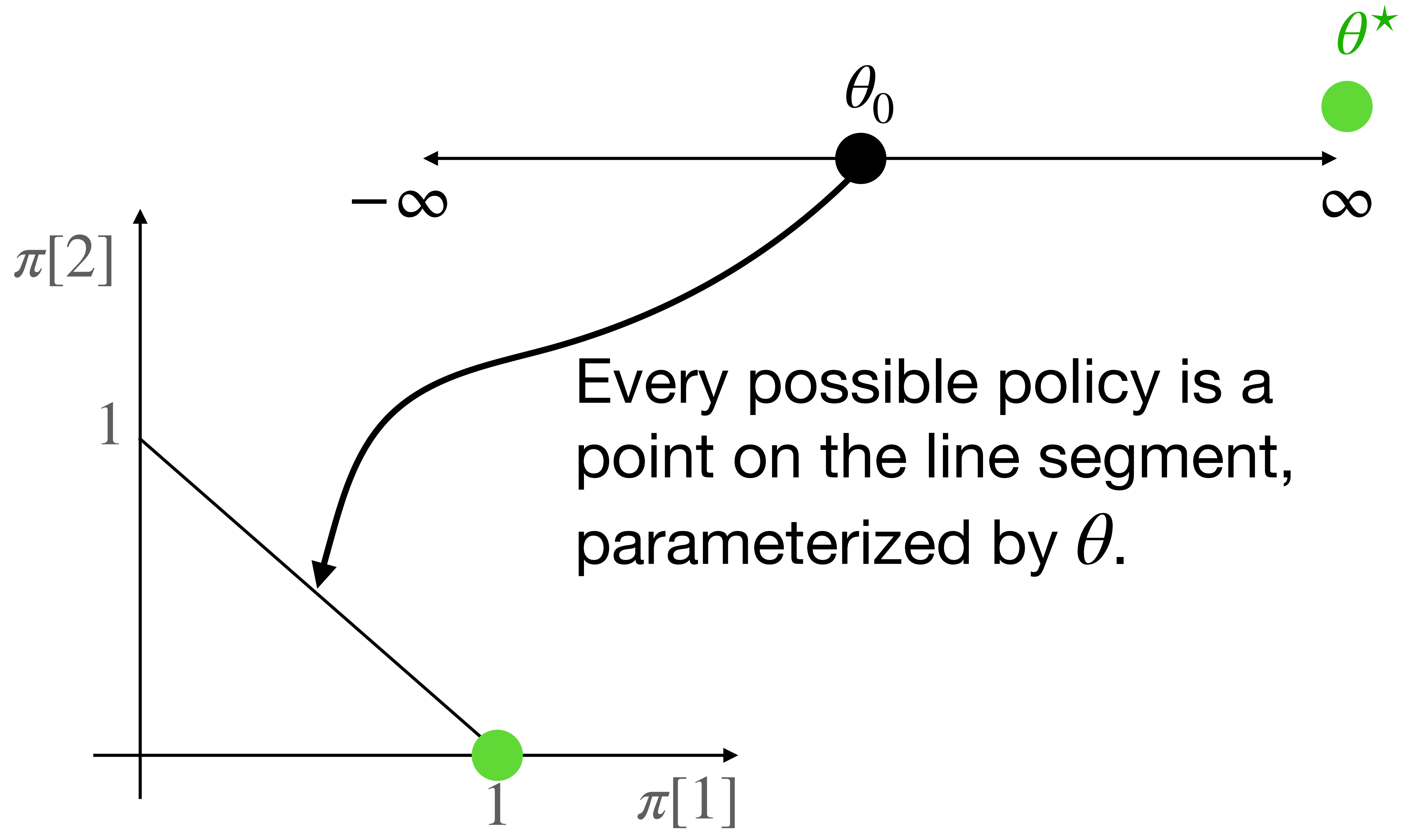

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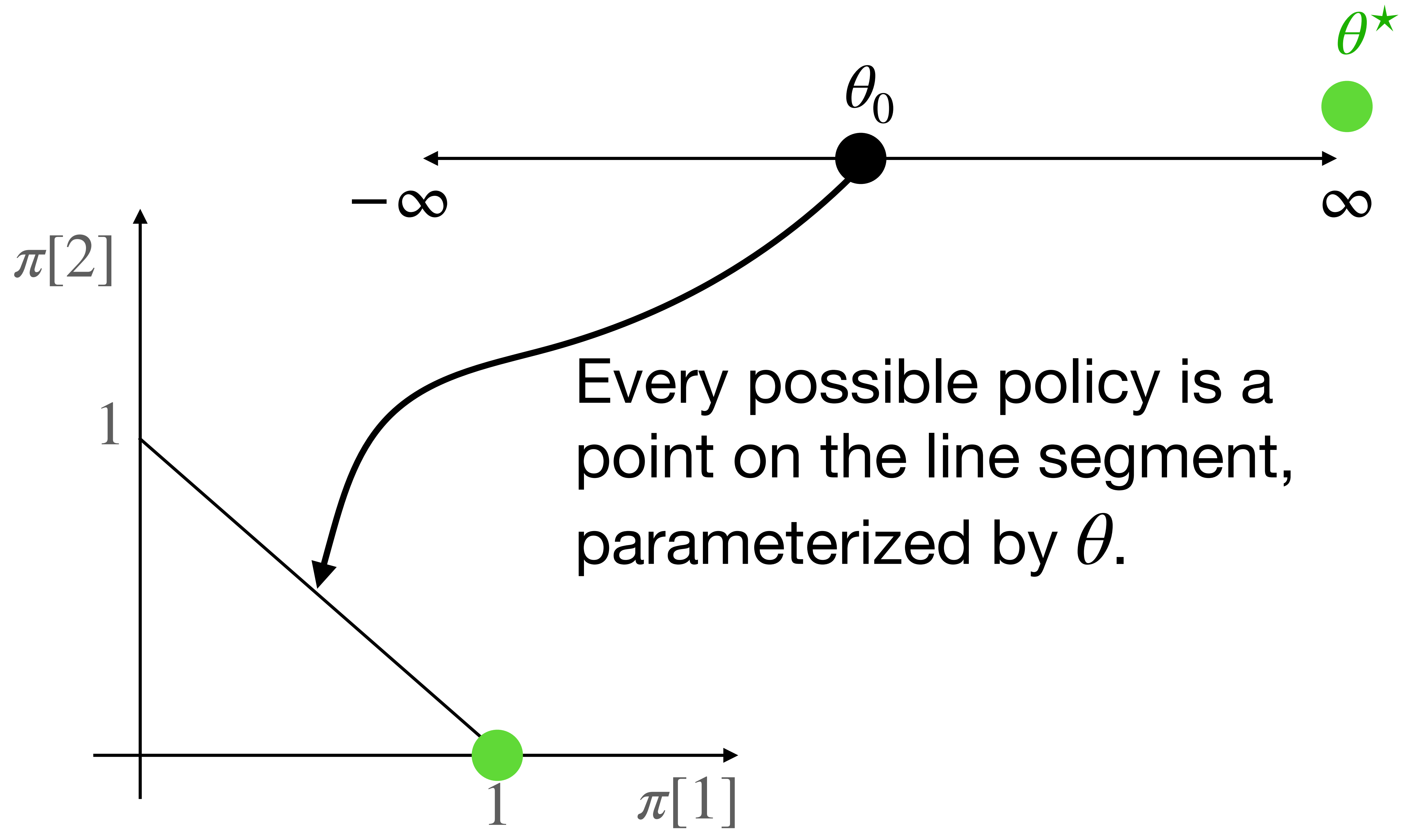

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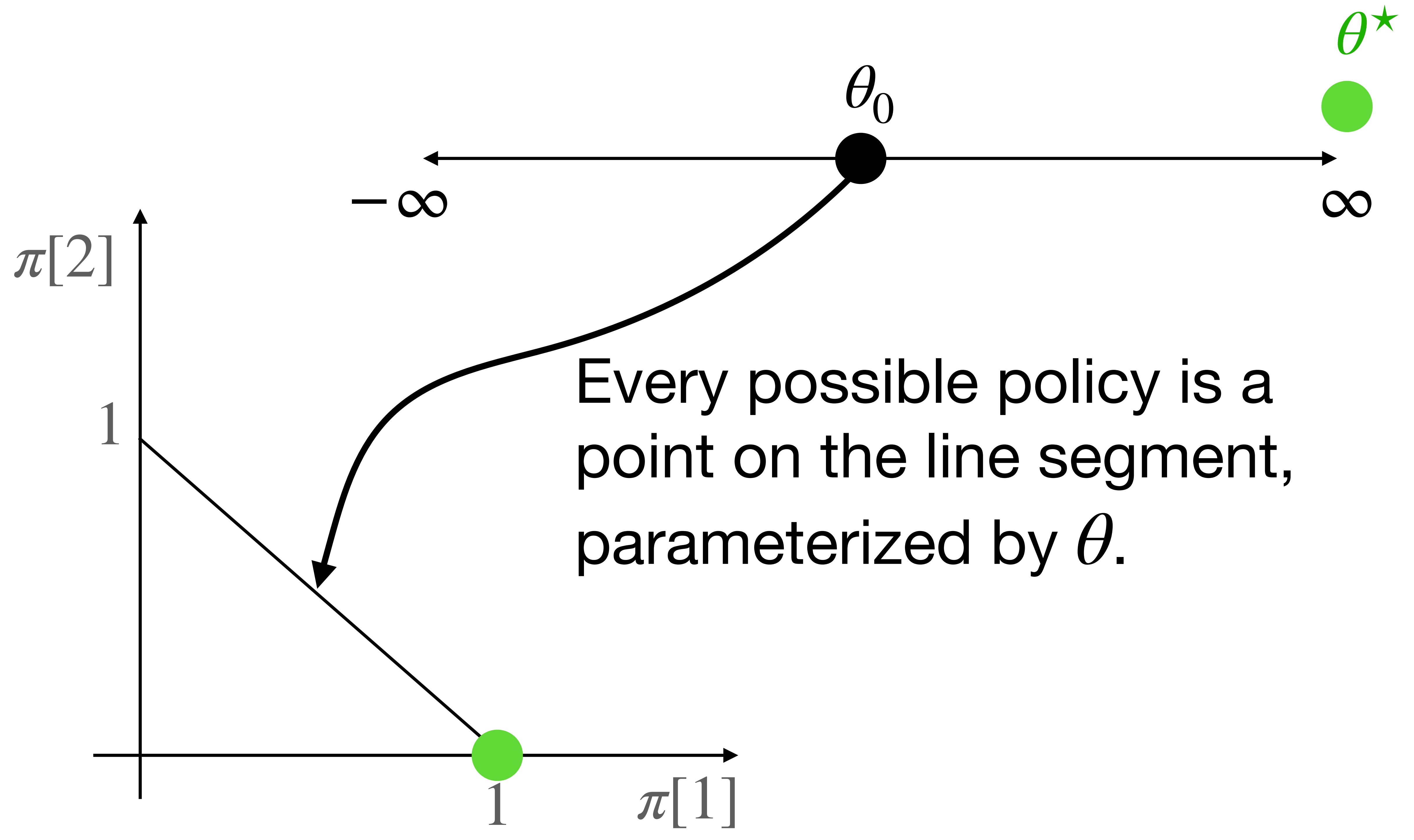

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i.e., vanilla GA moves to $\theta = \infty$ with smaller and smaller steps, since $\nabla_\theta J(\theta) \to 0$ as $\theta \to \infty$

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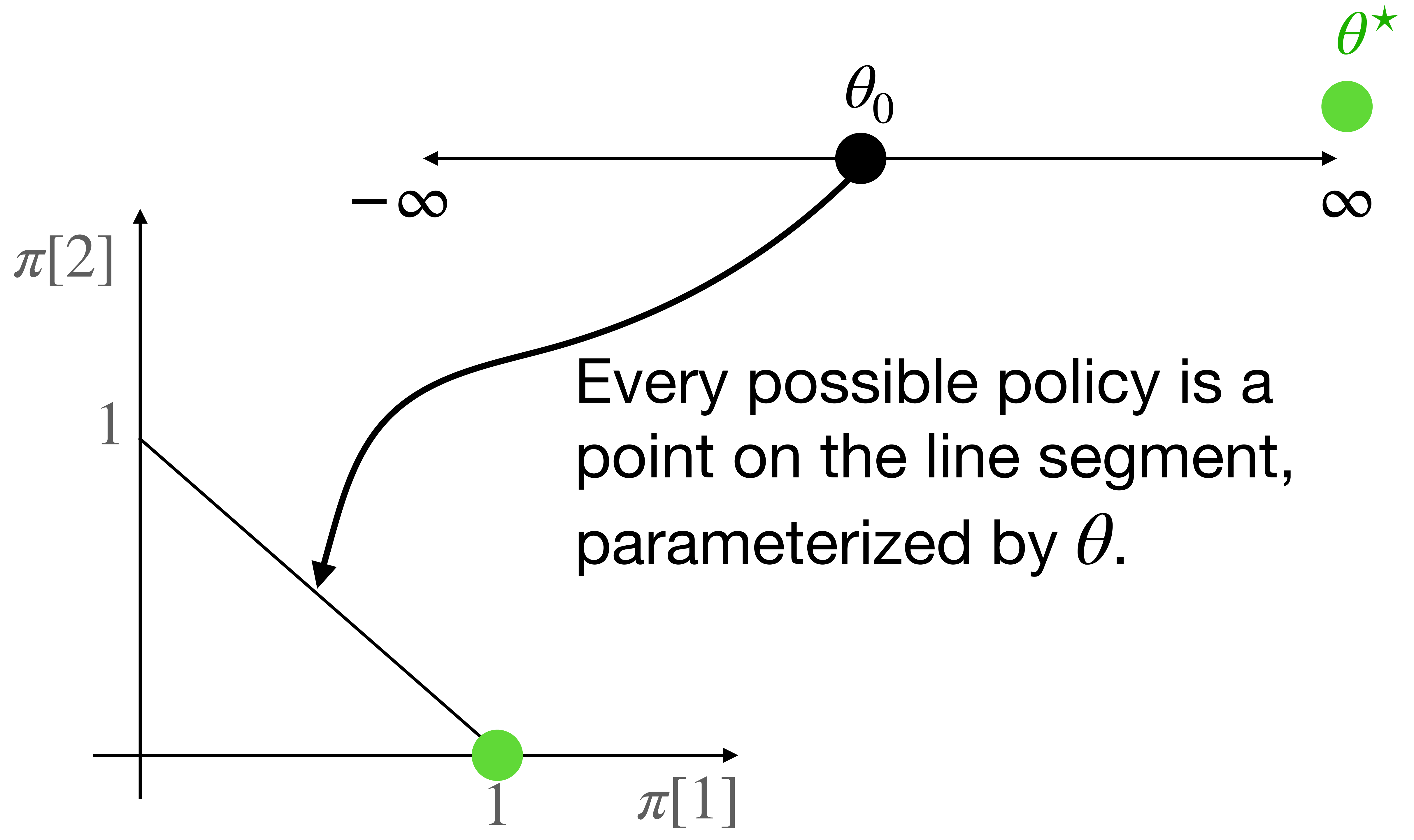

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i.e., vanilla GA moves to $\theta = \infty$ with smaller and smaller steps, since $\nabla_\theta J(\theta) \to 0$ as $\theta \to \infty$

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Example of Natural Gradient on 1-d problem: 2 actions, 1 state

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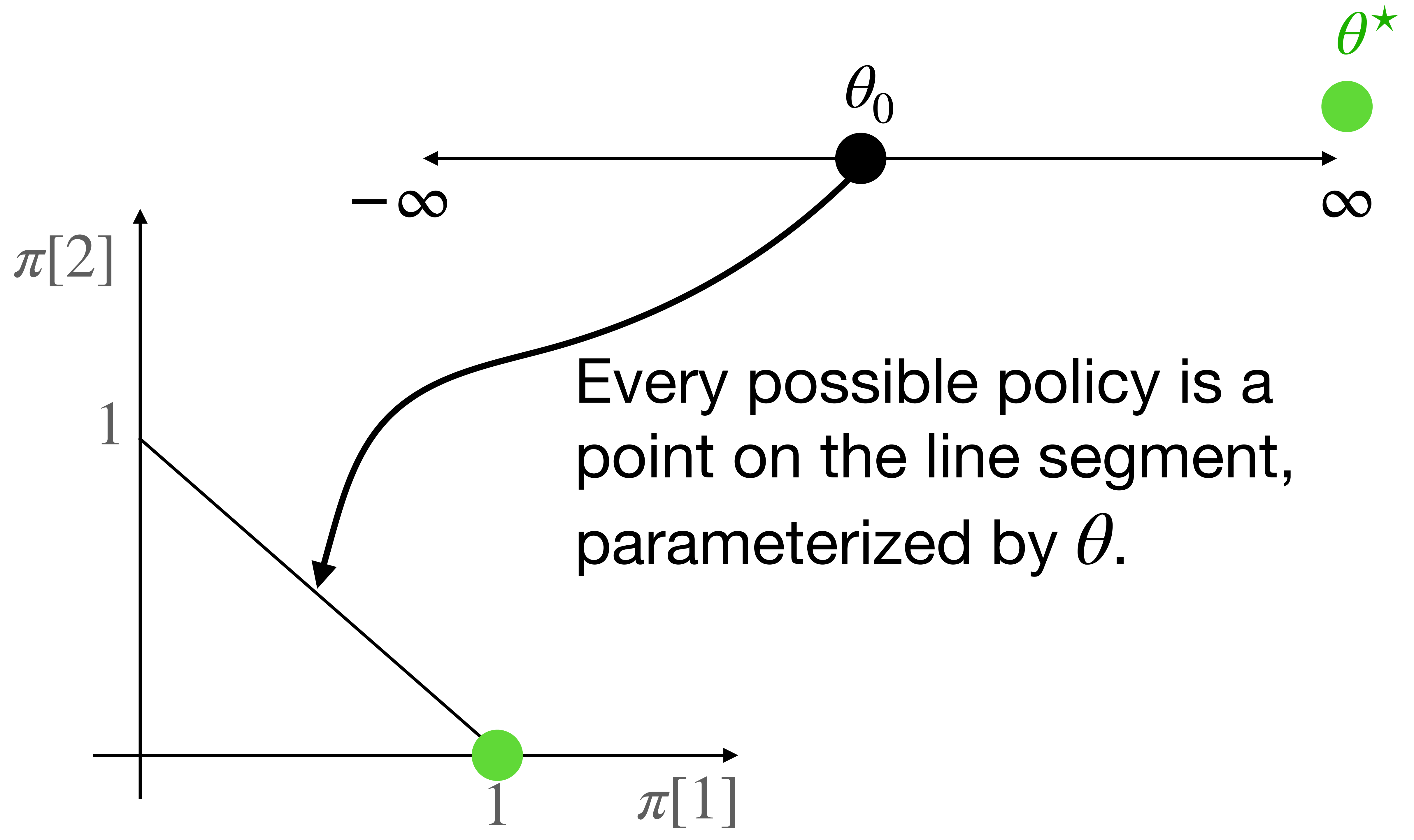

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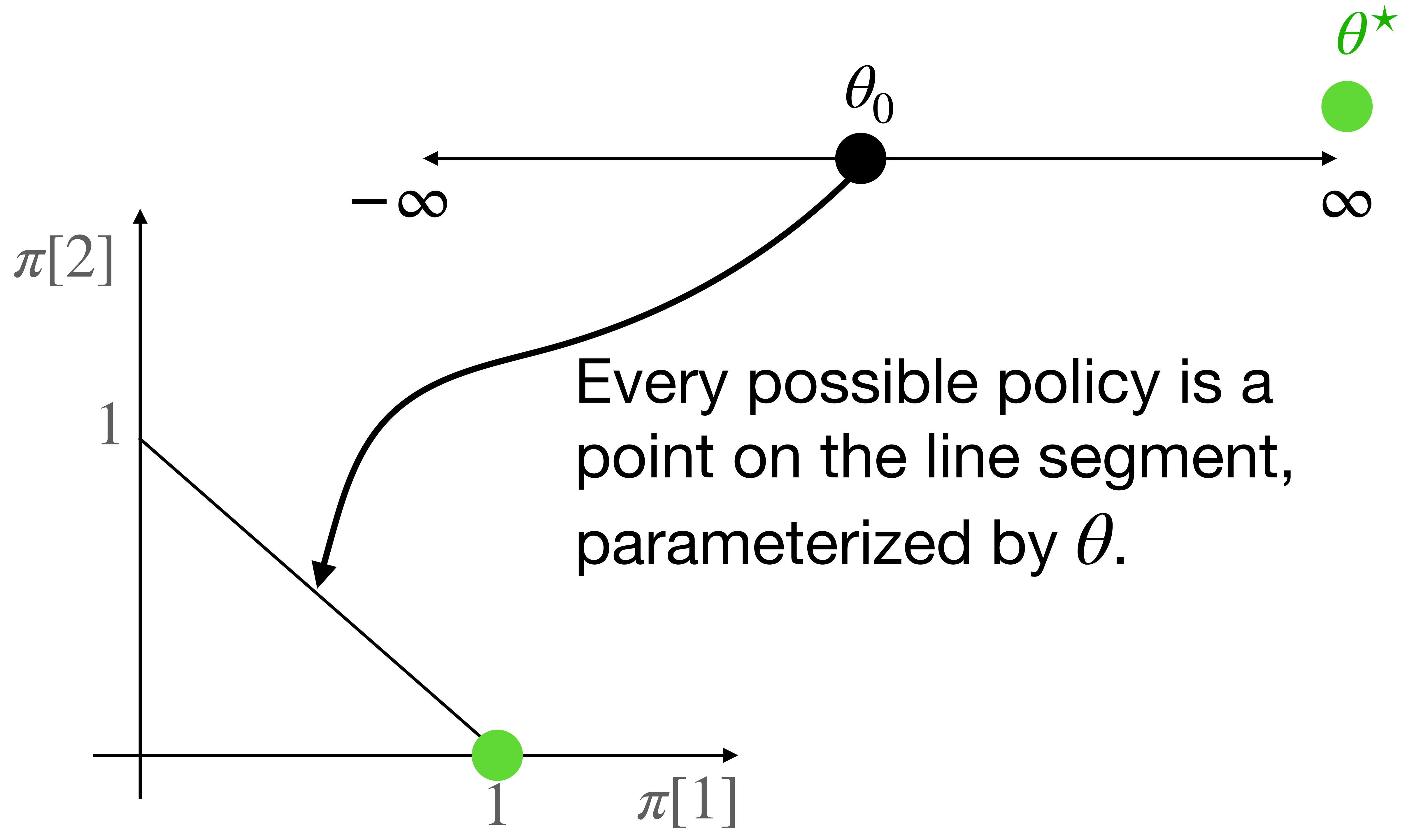

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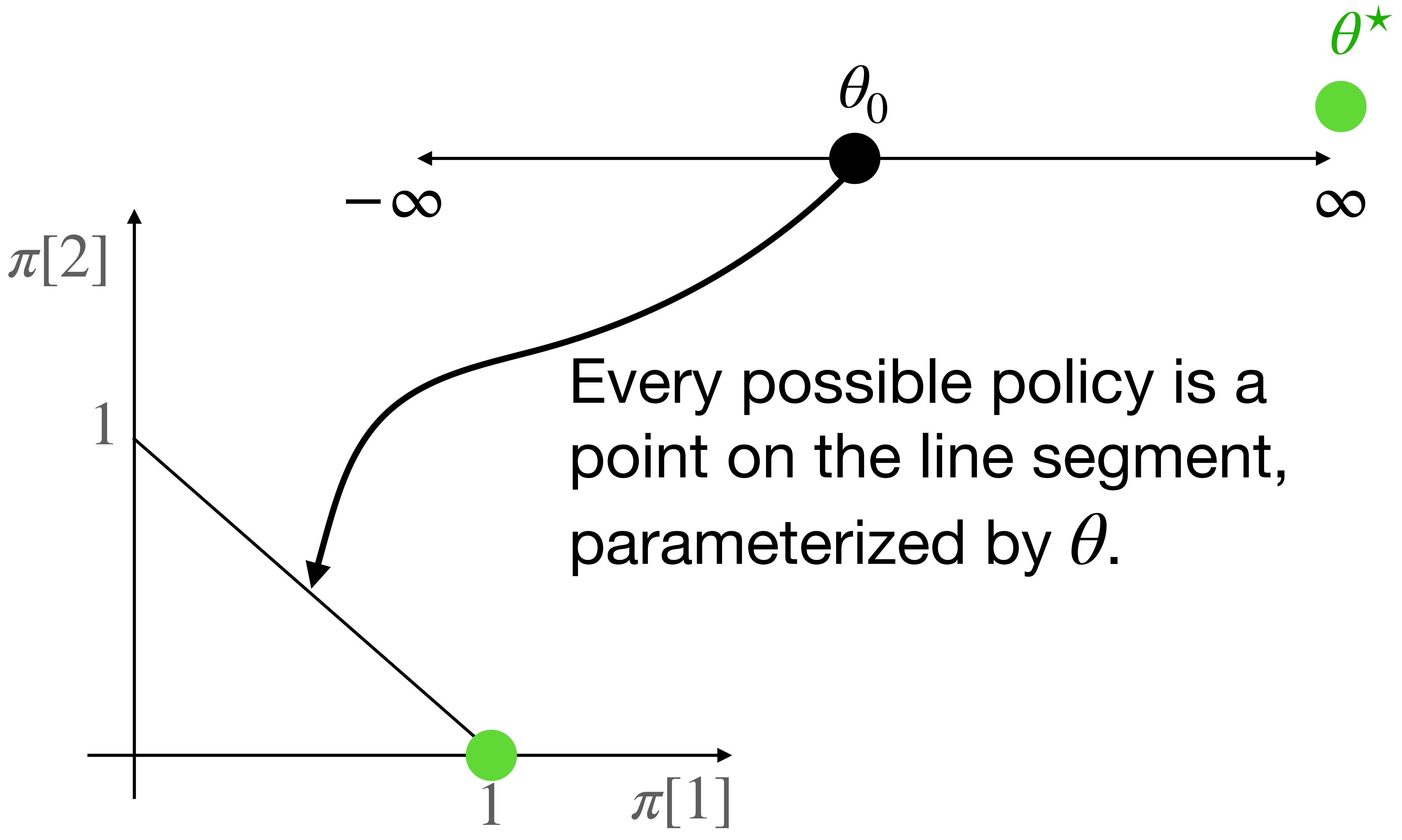

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NPG moves to $\theta = \infty$ much more quickly (for a fixed η)