Trust Region Policy Optimization & The Natural Policy Gradient

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CS/Stat 184(0): Introduction to Reinforcement Learning

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Today

- Feedback from last lecture
- Recap
- The Performance Difference Lemma
- Trust Region Policy Optimization (TRPO)
- The Natural Policy Gradient (NPG)

Feedback from feedback forms

1. Thank you to everyone who filled out the forms!
2. Discuss projects!

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Optimization Objective

- Consider a parameterized class of policies:

$$\{\pi_\theta(a \mid s) \mid \theta \in \mathbb{R}^d\}$$

  (why do we make it stochastic?)

- Objective $\max_\theta J(\theta)$, where

$$J(\theta) := \mathbb{E}_{s_0 \sim \mu}\left[V^{\pi_\theta}(s_0)\right] = \mathbb{E}_{\tau \sim \rho_{\pi_\theta}}\left[\sum_{h=0}^{H-1} r(s_h, a_h)\right]$$

- Policy Gradient Descent:

$$\theta^{k+1} = \theta^k + \eta \nabla J(\theta^k)$$

REINFORCE: A Policy Gradient Algorithm

- Let $\rho_\theta(\tau)$ be the probability of a trajectory $\tau = \{s_0, a_0, s_1, a_1, \ldots, s_{H-1}, a_{H-1}\}$, i.e.

$$\rho_\theta(\tau) = \mu(s_0)\pi_\theta(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\ldots P(s_{H-1} \mid s_{H-2}, a_{H-2})\pi_\theta(a_{H-1} \mid s_{H-1})$$

- Let $R(\tau)$ be the cumulative reward on trajectory τ, i.e. $R(\tau) := \sum_{h=0}^{H-1} r(s_h, a_h)$

- Our objective function is:

$$J(\theta) \ = E_{\tau\sim\rho_\theta}\Big[R(\tau)\Big]$$

- From the likelihood ratio method, we have:

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau\sim\rho_\theta}\left[\nabla_\theta \ln \rho_\theta(\tau)\ R(\tau)\right]$$

- The REINFORCE Policy Gradient expression:

$$\nabla_\theta \ln \rho_\theta(\tau)\ R(\tau) = \left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)$$

Obtaining an Unbiased Gradient Estimate at θ

$$\nabla_\theta J(\theta) := \mathbb{E}_{\tau \sim \rho_\theta}\left[\left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)\right]$$

1. Obtain a trajectory $\tau \sim \rho_\theta$
   (which we can do in our learning setting)
2. Set:

$$g(\theta, \tau) := \left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)$$

We have: $\mathbb{E}[g(\theta, \tau)] = \nabla_\theta J(\theta)$

PG with REINFORCE:

1. Initialize θ^0, step size parameters: $\eta^1, \eta^2, \ldots$
2. For $k = 0, \ldots$:
    1. Obtain a trajectory $\tau \sim \rho_{\theta^k}$
       Compute $g(\theta^k, \tau)$
    2. Update: $\theta^{k+1} = \theta^k + \eta^k g(\theta^k, \tau)$

Other PG formulas
(that are lower variance for sampling)

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta}\left[\left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)\right] \qquad \text{(REINFORCE)}$$

$$= \mathbb{E}_{\tau \sim \rho_\theta}\left[\sum_{h=0}^{H-1}\left(\nabla_\theta \ln \pi_\theta(a_h \mid s_h) \sum_{t=h}^{H-1} r(s_t, a_t)\right)\right]$$

$$= \mathbb{E}_{\tau \sim \rho_\theta}\left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) Q_h^{\pi_\theta}(s_h, a_h)\right]$$

Intuition: Changing the action distribution at h only affects rewards later on…

HW: You will show these simplified version are also valid PG expressions

With a "baseline" function:

For any function only of the state, $b_h : S \to \mathbb{R}$, we have:

$$\begin{aligned}\nabla_\theta J(\theta) &= \mathbb{E}_{\tau \sim \rho_\theta}\left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \,|\, s_h)\Big(R_h(\tau) - b_h(s_h)\Big)\right] \\ &= \mathbb{E}_{\tau \sim \rho_\theta}\left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \,|\, s_h)\left(Q_h^{\pi_\theta}(s_h, a_h) - b_h(s_h)\right)\right]\end{aligned}$$

This is (basically) the method of control variates.

- For the proof, it was helpful to note:
  $\mathbb{E}_{x \sim P_\theta}\left[\nabla_\theta \log P_\theta(x)\ c\right] = 0$

The Advantage Function (finite horizon)

$$V_h^\pi(s) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \middle| s_h = s\right] \qquad Q_h^\pi(s,a) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \middle| (s_h, a_h) = (s,a)\right]$$

- The Advantage function is defined as:

$$A_h^\pi(s,a) = Q_h^\pi(s,a) - V_h^\pi(s)$$

- We have that:

$$\mathbb{E}_{a\sim\pi(\cdot|s)}\left[A_h^\pi(s,a) \middle| s, h\right] = \sum_a \pi(a\,|\,s) A_h^\pi(s,a) = 0$$

- We know $A_h^{\pi^\star}(s,a) \leq 0 \quad \forall s, a$

- For the discounted case, $A^\pi(s,a) = Q^\pi(s,a) - V^\pi(s)$

The Advantage-based PG:

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta(\tau)} \left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) \left(Q_h^{\pi_\theta}(s_h, a_h) - b_h(s_h) \right) \right]$$

$$= \mathbb{E}_{\tau \sim \rho_\theta(\tau)} \left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) A_h^{\pi_\theta}(s_h, a_h) \right]$$

- The second step follows by choosing $b_h(s) = V_h^\pi(s)$.
- In practice, the most common approach is to use $b_h(s)$ that's an estimate of $V_h^\pi(s)$.

PG with a Learned Baseline:

Let $g'(\theta, \tau, b()) := \sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h | s_h)\big(R_h(\tau) - b(s_h, h)\big)$

1. Initialize θ^0, parameters: $\eta^1, \eta^2, \ldots$
2. For $k = 0, \ldots$:
    1. Supervised Learning: Using N trajectories sampled under π_{θ^k}, estimate a baseline $\widetilde{b}$
       $\widetilde{b}(s, h) \approx V_h^{\theta^k}(s)$
    2. Obtain a trajectory $\tau \sim \rho_{\theta^k}$
       Compute $g'(\theta^k, \tau, \widetilde{b}())$
    3. Update: $\theta^{k+1} = \theta^k + \eta^k g'(\theta^k, \tau, \widetilde{b}())$

Note that regardless of our choice of $\widetilde{b}$, we still get unbiased gradient estimates.

(minibatch) PG with a Learned Baseline:

1. Initialize θ^0, parameters: $\eta^1, \eta^2, \ldots$
2. For $k = 0, \ldots$:
   1. Supervised Learning: Using N trajectories sampled under π_{θ^k}, estimate a baseline $\widetilde{b}$
      $\widetilde{b}(s, h) \approx V_h^{\theta^k}(s)$
   2. Obtain M trajectories $\tau_1, \ldots \tau_M \sim \rho_{\theta^k}$

      Compute $g = \frac{1}{M} \sum_{m=1}^{M} g'(\theta^k, \tau_m, \widetilde{b}())$

   3. Update: $\theta^{k+1} = \theta^k + \eta^k g$

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Recall: Fitted Policy Iteration

- Initialization: choose a policy $\pi^0 : S \mapsto A$ and a sample size N
- For $k = 0,1,\ldots$
  1. <u>Fitted</u> **Policy Evaluation**: Using N sampled trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi^k}$, obtain approximation $\hat{Q}^{\pi^k} \approx Q^{\pi^k}$
  2. **Policy Improvement**: set $\pi_h^{k+1}(s) := \arg\max_a \hat{Q}^{\pi^k}(s, a, h)$

Fitted Policy Iteration: Advantage Version

- Initialization: choose a policy $\pi^0 : S \mapsto A$ and a sample size N
- For $k = 0,1,\ldots$
  1. **Fitted Policy Evaluation**: Using N sampled trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi^k}$, obtain approximation $\hat{A}^{\pi^k} \approx A^{\pi^k}$
  2. **Policy Improvement**: set $\pi_h^{k+1}(s) := \arg\max_a \hat{A}^{\pi^k}(s, a, h)$

The Performance Difference Lemma (PDL)

- Let $\rho_{\widetilde{\pi},s}$ be the distribution of trajectories from starting state s acting under $\widetilde{\pi}$. (we are making the starting distribution explicit now).
- For any two policies π and $\widetilde{\pi}$ and any state s,

$$V^{\widetilde{\pi}}(s) - V^{\pi}(s) = \mathbb{E}_{\tau \sim \rho_{\widetilde{\pi},s}} \left[\sum_{h=0}^{H-1} A^{\pi}(s_h, a_h, h) \right]$$

Comments:

- Helps us think about error analysis, instabilities of fitted PI, and sub-optimality.
- Helps to understand algorithm design (TRPO, NPG, PPO)
- This also motivates the use of “local” methods (e.g. policy gradient descent)

Back to Fitted Policy Iteration

- Suppose π^k gets updated to π^{k+1}. How much worse could π^{k+1} be?
- In Fitted Policy Iteration, $\hat{A}^{\pi^k} \approx A^{\pi^k}$ is achieved via supervised learning on $\tau_1, \ldots \tau_N \sim \rho_{\pi^k}$
- This means we expect $\mathbb{E}_{\tau \sim \rho_{\pi^k,s}}\left[\sum_{h=0}^{H-1} \hat{A}^{\pi^k}(s_h, a_h, h)\right] \approx \mathbb{E}_{\tau \sim \rho_{\pi^k,s}}\left[\sum_{h=0}^{H-1} A^{\pi^k}(s_h, a_h, h)\right]$
- In particular, $\hat{A}^{\pi^k}$ should be close to A^{π^k} where π^k visits often…
- But it could be very bad in places π^k visits rarely, and nothing stops π^{k+1} from visiting those bad places very often!
- So π^{k+1} could end up being (much) worse than π^k

- Problem is a mismatch between expectations: what we really want is

$$\mathbb{E}_{\tau \sim \rho_{\pi^{k+1},s}}\left[\sum_{h=0}^{H-1} \hat{A}^{\pi^k}(s_h, a_h, h)\right] \approx \mathbb{E}_{\tau \sim \rho_{\pi^{k+1},s}}\left[\sum_{h=0}^{H-1} A^{\pi^k}(s_h, a_h, h)\right]$$

- One way to ensure this: keep $\pi^{k+1} \approx \pi^k$

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A trust region formulation for policy update:

- What's bad about fitted PI?
  even if we pick better actions "on average", the trajectory updates are unstable
- Can we fix this?
  Let's look at an incremental policy updating approach

1. Initialize θ^0
2. For $k = 0, \ldots, K$:
   try to approximately solve:

$$\theta^{k+1} = \arg\max_{\theta} \mathbb{E}_{s_0,\ldots,s_{H-1}\sim\rho_{\pi_{\theta^k}}}\left[\sum_{h=0}^{H-1} \mathbb{E}_{a_h\sim\pi_\theta(\cdot|s_h)}\left[A^{\pi_{\theta^k}}(s_h, a_h, h)\right]\right]$$

s.t. ρ_θ is "close" to $\rho_{\pi_{\theta^k}}$

3. Return π_{θ^K}

- How should we define "close", i.e., what is our "trust region?

KL-divergence: measures the distance between two distributions

Given two distributions P & Q, where $P \in \Delta(X), Q \in \Delta(X)$,
KL Divergence is defined as:

$$KL(P|Q) = \mathbb{E}_{x \sim P}\left[\ln \frac{P(x)}{Q(x)}\right]$$

Examples:

If $Q = P$, then $KL(P|Q) = KL(Q|P) = 0$

If $P = \mathcal{N}(\mu_1, \sigma^2 I), Q = \mathcal{N}(\mu_2, \sigma^2 I)$, then $KL(P|Q) = \frac{1}{2\sigma^2}\|\mu_1 - \mu_2\|^2$

Fact:

$KL(P|Q) \geq 0$, and is 0 if and only if $P = Q$

Trust Region Policy Optimization (TRPO)

1. Initialize θ^0
2. For $k = 0, \ldots, K$:
   try to approximately solve:

$$\theta^{k+1} = \arg\max_{\theta} \mathbb{E}_{s_0,\ldots,s_{H-1}\sim\rho_{\pi_{\theta^k}}} \left[\sum_{h=0}^{H-1} \mathbb{E}_{a_h\sim\pi_\theta(\cdot|s_h)} \left[A^{\pi_{\theta^k}}(s_h, a_h, h) \right] \right]$$

$$\text{s.t. } KL\left(\rho_{\pi_{\theta^k}} | \rho_{\pi_\theta}\right) \leq \delta$$

3. Return π_{θ^K}

- We want to maximize local advantage against π_{θ^k},
  but we want the new policy to be close to π_{θ^k} (in the KL sense)
- How do we implement this with sampled trajectories?)

How do we implement TRPO with samples?

1. Initialize parameter θ^0, sample size M, and tolerance δ
2. For $k = 0,\ldots,K$:
    1. [Advantage-Evaluation Subroutine]
       Using M sampled trajectories $\tau_1, \ldots \tau_M \sim \rho_{\pi_{\theta^k}}$, obtain approximation $\hat{A}^{\pi_{\theta^k}} \approx A^{\pi_{\theta^k}}$
    2. Solve the following optimization problem to obtain θ^{k+1}:

$$\max_{\theta} \sum_{m=1}^{M} \sum_{h=0}^{H-1} \mathbb{E}_{a\sim\pi_\theta(\cdot|s_h^m)} \left[\hat{A}^{\pi_{\theta^k}}(s_h^m, a, h)\right]$$

$$\text{s.t.} \sum_{m=1}^{M} \sum_{h=0}^{H-1} \ln \frac{\pi_{\theta^k}(a_h^m \mid s_h^m)}{\pi_\theta(a_h^m \mid s_h^m)} \leq \delta$$

Approximate expectation by importance sampling:

$$\mathbb{E}_{a\sim\pi_\theta(\cdot|s_h^m)} \left[\hat{A}^{\pi_{\theta^k}}(s_h^m, a, h)\right]$$

$$= \mathbb{E}_{a\sim\pi_{\theta^k}(\cdot|s_h^m)} \left[\frac{\pi_\theta(a \mid s_h^m)}{\pi_{\theta^k}(a \mid s_h^m)} \hat{A}^{\pi_{\theta^k}}(s_h^m, a, h)\right]$$

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TRPO is locally equivalent to a much simpler algorithm

TRPO at iteration k:

$$\max_\theta \mathbb{E}_{s_0,\ldots,s_{H-1}\sim\rho_{\pi_{\theta^k}}}\left[\sum_{h=0}^{H-1}\mathbb{E}_{a_h\sim\pi_\theta(\cdot|s_h)}\left[A^{\pi_{\theta^k}}(s_h,a_h,h)\right]\right] \longrightarrow \text{First-order Taylor expansion at } \theta^k$$

$$\text{s.t. } KL\left(\rho_{\pi_{\theta^k}} | \rho_{\pi_\theta}\right) \le \delta \longrightarrow \text{second-order Taylor expansion at } \theta^k$$

Intuition: maximize local advantage subject to being incremental (in KL)

$$\max_\theta \nabla_\theta J(\theta^k)^\top(\theta-\theta^k)$$

$$\text{s.t. } (\theta-\theta^k)^\top F_{\theta^k}(\theta-\theta^k) \le \delta$$

(Where F_{θ^k} is the “Fisher Information Matrix”)

Natural Policy Gradient (NPG): A “leading order” equivalent program to TRPO:

1. Initialize θ^0
2. For $k = 0, \ldots, K$:
$$\theta^{k+1} = \arg\max_{\theta} \nabla_\theta J(\theta^k)^\top (\theta - \theta^k)$$
$$\text{s.t. } (\theta - \theta^k)^\top F_{\theta^k} (\theta - \theta^k) \leq \delta$$
3. Return π_{θ^K}

- Where $\nabla_\theta J(\theta^k)$ is the gradient of $J(\theta)$ evaluated at θ^k, and
- F_θ is (basically) the Fisher information matrix at $\theta \in \mathbb{R}^d$, defined as:

$$F_\theta := \mathbb{E}_{\tau \sim \rho_{\pi_\theta}} \left[\nabla_\theta \ln \rho_\theta(\tau) \left(\nabla_\theta \ln \rho_\theta(\tau) \right)^\top \right] \in \mathbb{R}^{d \times d}$$

$$= \mathbb{E}_{\tau \sim \rho_{\pi_\theta}} \left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) \left(\nabla_\theta \ln \pi_\theta(a_h \mid s_h) \right)^\top \right]$$

NPG has a closed form update!

1. Initialize θ^0
2. For $k = 0, \ldots, K$:
$$\theta^{k+1} = \arg\max_{\theta} \nabla_\theta J(\theta^k)^\top (\theta - \theta^k)$$
s.t. $(\theta - \theta^k)^\top F_{\theta^k} (\theta - \theta^k) \leq \delta$
3. Return π_{θ^K}

Linear objective and quadratic convex constraint: we can solve it optimally!

Indeed this gives us:

$$\theta^{k+1} = \theta^k + \eta F_{\theta^k}^{-1} \nabla_\theta J(\theta^k)$$

Where $\eta = \sqrt{\dfrac{\delta}{\nabla_\theta J(\theta^k)^\top F_{\theta^k}^{-1} \nabla_\theta J(\theta^k)}}$

An Implementation: Sample Based NPG

1. Initialize θ^0
2. For $k = 0,\ldots,K$:
   - Obtain approximation of Policy Gradient: $\hat{g} \approx \nabla_\theta J(\theta^k)$
   - Obtain approximation of Fisher information: $\hat{F} \approx F_{\theta^k}$
   - Natural Gradient Ascent: $\theta^{k+1} = \theta^k + \eta \hat{F}^{-1} \hat{g}$
3. Return π_{θ^K}

(We will implement it in HW4 on Cartpole)

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Summary:

1. Performance Difference Lemma tells us we need to stay local
2. TRPO and NPG ensure we don’t move too much each step

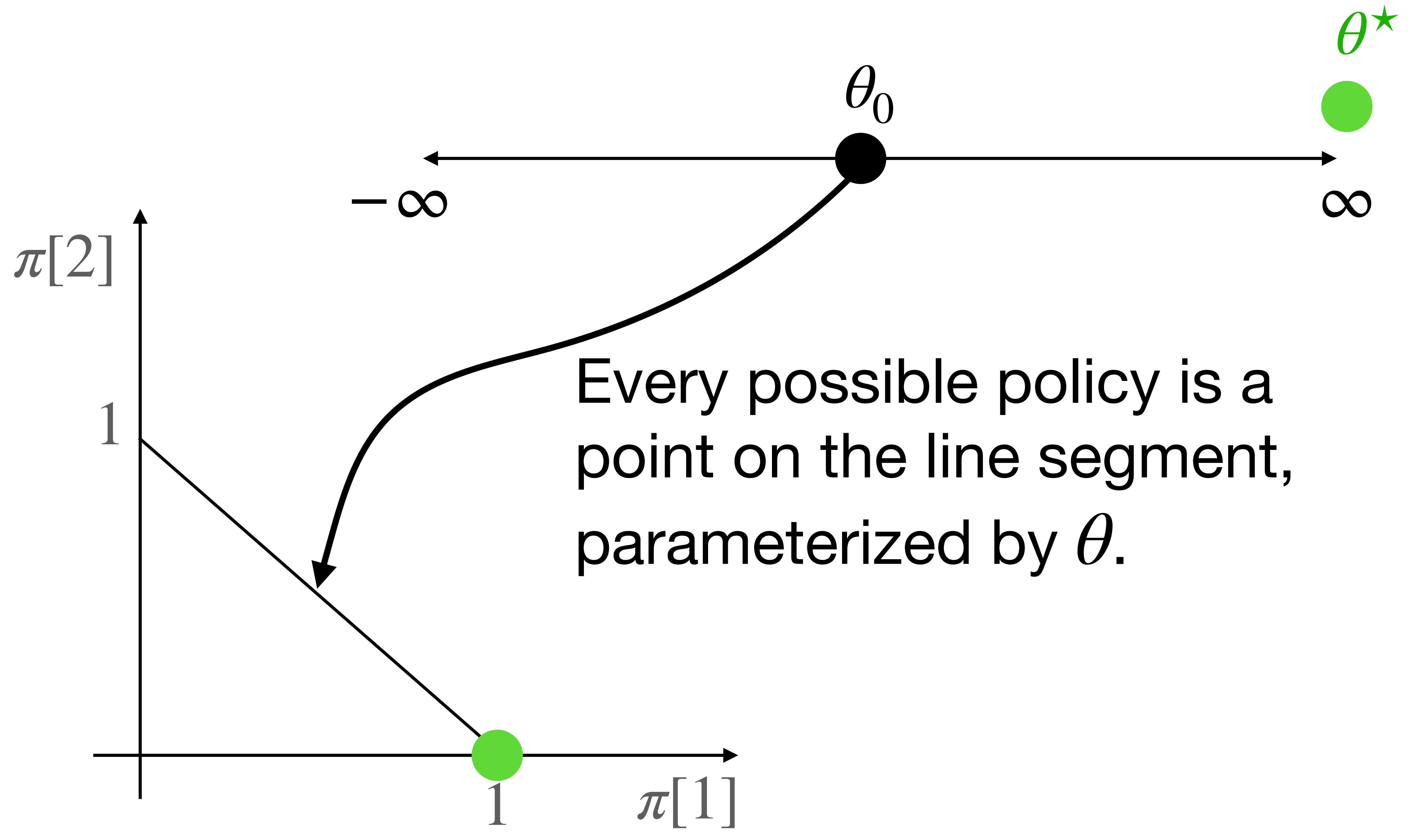
Attendance:
bit.ly/3RcTC9T

Feedback:
bit.ly/3RHtlxy

Example of Natural Gradient on 1-d problem: 2 actions, 1 state

$$(\pi_\theta[1], \pi_\theta[2]) := \left(\frac{\exp(\theta)}{1+\exp(\theta)}, \frac{1}{1+\exp(\theta)}\right)$$

$$J(\theta) = 100 \cdot \pi_\theta[1] + 1 \cdot \pi_\theta[2]$$



Gradient: $\nabla_\theta J(\theta) = \dfrac{99\exp(\theta)}{(1+\exp(\theta))^2}$

Exact PG: $\theta^{k+1} = \theta^k + \eta \dfrac{99\exp(\theta^k)}{(1+\exp(\theta^k))^2}$

i.e., vanilla GA moves to $\theta = \infty$ with smaller and smaller steps, since $\nabla_\theta J(\theta) \to 0$ as $\theta \to \infty$

Fisher information scalar: $F_\theta = \dfrac{\exp(\theta)}{(1+\exp(\theta))^2}$

NPG: $\theta^{k+1} = \theta^k + \eta \dfrac{\nabla_\theta J(\theta^k)}{F_{\theta^k}} = \theta_t + \eta \cdot 99$

NPG moves to $\theta = \infty$ much more quickly (for a fixed η)