# Policy Gradient Descent

**Lucas Janson**

**CS/Stat 184(0): Introduction to Reinforcement Learning**

**Fall 2024**

# Today

- Feedback from last lecture
- Recap+
- Gradient Descent (ok this is also sort of recap)
- Policy Gradient
- Likelihood ratio method

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Analogous Q-value DP, with same notational change as last lecture: $h$ as argument

1. Initialization: $Q(s,a,H) = 0 \quad \forall s,a$
2. Solve (via dynamic programming):

$$Q(s,a,h) = r(s,a) + \mathbb{E}_{s' \sim P(s,a)} \left[ \max_{a' \in A} Q(s',a',h+1) \right] \quad \forall s,a,h$$

3. Return:

$$\pi_h(s) = \arg\max_a \left\{ Q(s,a,h) \right\}$$

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Then we'd be happy if we found a

$$Q(s_h,a_h,h) = f(x) = \mathbb{E}[y \mid x] = \mathbb{E}\left[r(s_h,a_h) + \max_{a'} Q(s_{h+1},a',h+1) \,\middle|\, s_h,a_h,h\right]$$

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Setting that aside for the moment, to fit supervised learning, we'd minimize a least-squares objective function: $\hat{f}(x) = \arg\min_{f \in \mathcal{F}} \sum_{i=1}^{NH} (y_i - f(x_i))^2$

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Then if we have enough data, choose a good $\mathcal{F}$, and optimize well,

$$Q(s_h, a_h, h) := \hat{f}(x) \approx \mathbb{E}[y \mid x] = \mathbb{E}\left[ r(s_h, a_h) + \max_{a'} Q(s_{h+1}, a', h+1) \middle| s_h, a_h, h \right]$$

# Fitted (Q-)Value Iteration

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1. Initialize fitted $Q$ function at $f_0$
2. For $k = 1, \ldots, K$:

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Q-Learning is an online version, i.e., draw new trajectories at each $k$ based on $f_k$ as $Q$-function

# Bonus: Q-learning

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$$Q(s_h, a_h, h) \leftarrow Q(s_h, a_h, h) - \eta \left( Q(s_h, a_h, h) - r(s_h, a_h) - \max_a Q(s_{h+1}, a, h + 1) \right)$$

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- Q-learning is an "off-policy" algorithm.
- Guarantee: Assuming states, actions visited infinitely often (which can be guaranteed with the action policy), $Q \rightarrow Q^\star$.

# Q-Learning with Function Approximation (extra material: read later if interested)

- Init: $Q(s, a, h)$
- For $k = 1,2,\ldots K$ episodes
  - Within each episode, for $h = 0,1,\ldots H-1$
    - Act: choose actions however you like!
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    - Update:
      $$\theta \leftarrow \theta - \eta\Big(f_\theta(s_h, a_h, h) - r(s_h, a_h) - \max_a f_\theta(s_{h+1}, a, h+1)\Big)\nabla f_\theta(s_h, a_h, h)$$
- Return $Q(s, a, h)$

- How to understand this expression?
  Consider doing a small step of SGD on the fitted-Q objective function.

# Recall: Policy Iteration (PI)

- Initialization: choose a policy $\pi^0 : S \mapsto A$
- For $k = 0,1,\ldots$
  1. **Policy Evaluation**: Solve (via dynamic programming):
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1. Initialize fitted $Q^\pi$ function at $f_0$
2. For $k = 1, \ldots, K$:

$$f_k = \arg\min_{f \in \mathcal{F}} \sum_{i=1}^{N} \sum_{h=1}^{H-1} \left( f(s_h^i, a_h^i, h) - \left( r(s_h^i, a_h^i) + f_{k-1}(s_{h+1}^i, \pi(s_h^i), h+1) \right) \right)^2$$

3. Return the function $f_K$ as an estimate of $Q^\pi$

# Fitted Policy Iteration:

- Initialization: choose a policy $\pi^0 : S \mapsto A$ and a sample size $N$
- For $k = 0,1,\ldots$
  1. **Fitted Policy Evaluation**: Using $N$ sampled trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi^k}$, obtain approximation $\hat{Q}^{\pi^k} \approx Q^{\pi^k}$
  2. **Policy Improvement**: set $\pi_h^{k+1}(s) := \arg\max_a \hat{Q}^{\pi^k}(s, a, h)$

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Using the definition of the $Q$ function, can do a non-iterative fitted policy evaluation

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Return:

$$\hat{Q}^\pi = \arg\min_{f \in \mathscr{F}} \sum_{i=1}^{N} \sum_{h=1}^{H-1} \left(f(s_h^i, a_h^i, h) - \sum_{t=h}^{H-1} r(s_t^i, a_t^i)\right)^2$$

# Bonus: TD(0)

(see posted slides)

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    - Act according to $\pi$

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$\hat{Q} = f_\theta$

(same for Q-learning w/ FA)

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- ✓ Feedback from last lecture
- ✓ Recap+
- Gradient Descent (ok this is also sort of recap)
- Policy Gradient
- Likelihood ratio method

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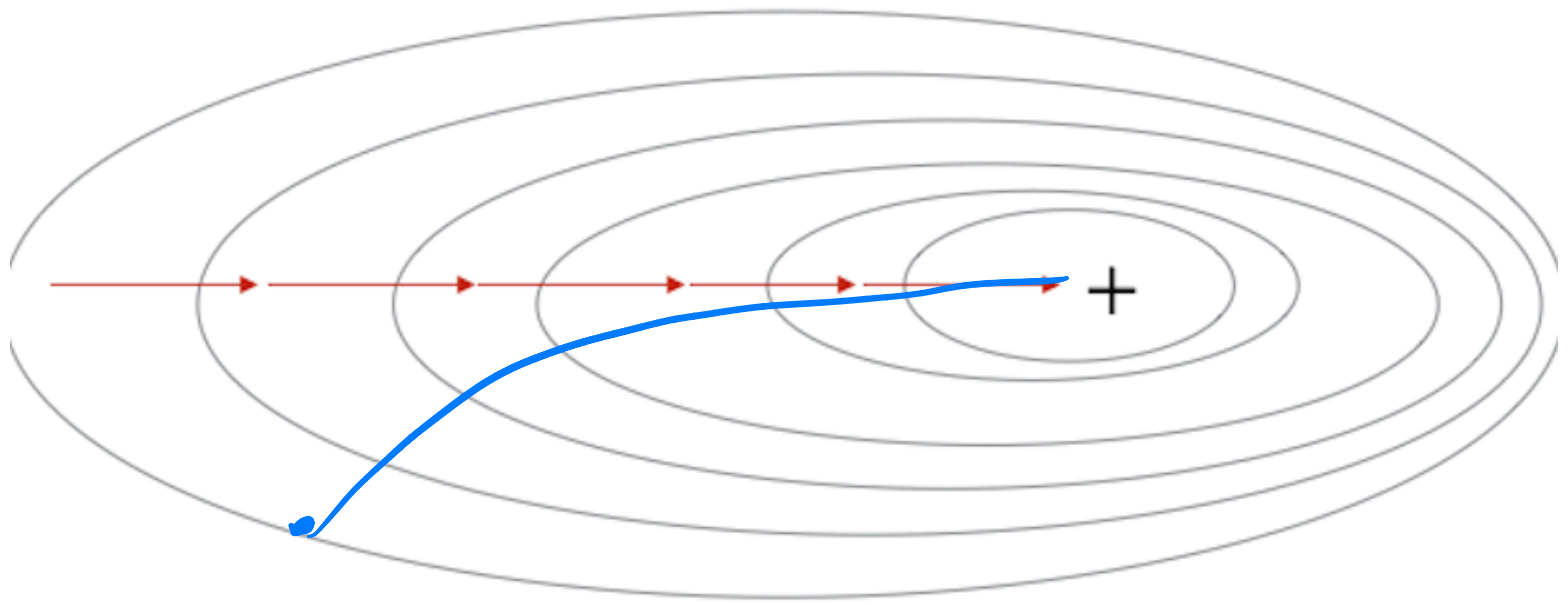

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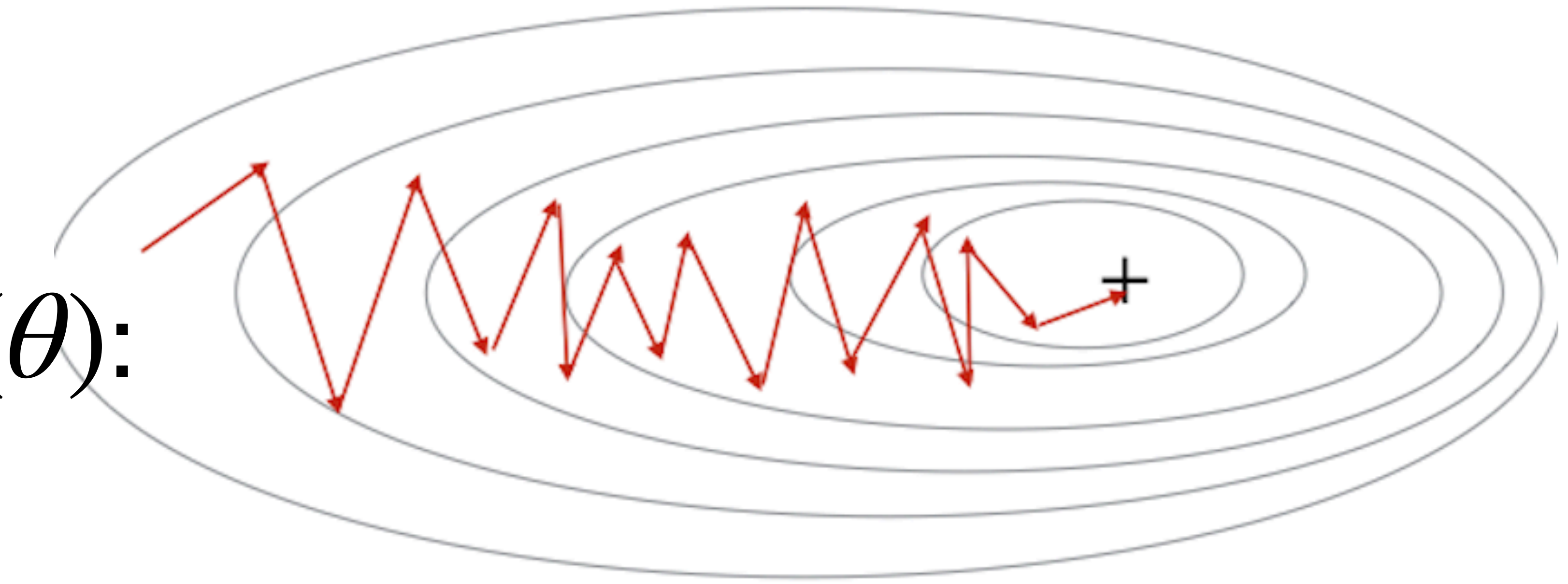
Stochastic Gradient Descent



- Stochastic Gradient Descent uses (unbiased) estimates of $\nabla J(\theta)$:
  - Initialize $\theta^0$, for $k = 0, \ldots$ :

$$\theta^{k+1} = \theta^k - \eta^k g^k, \quad \text{where } \mathbb{E}[g^k] = \nabla_\theta J(\theta^k)$$

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- Note with $\eta = 1$, we find the optima, $\theta^\star = c$, in one step.

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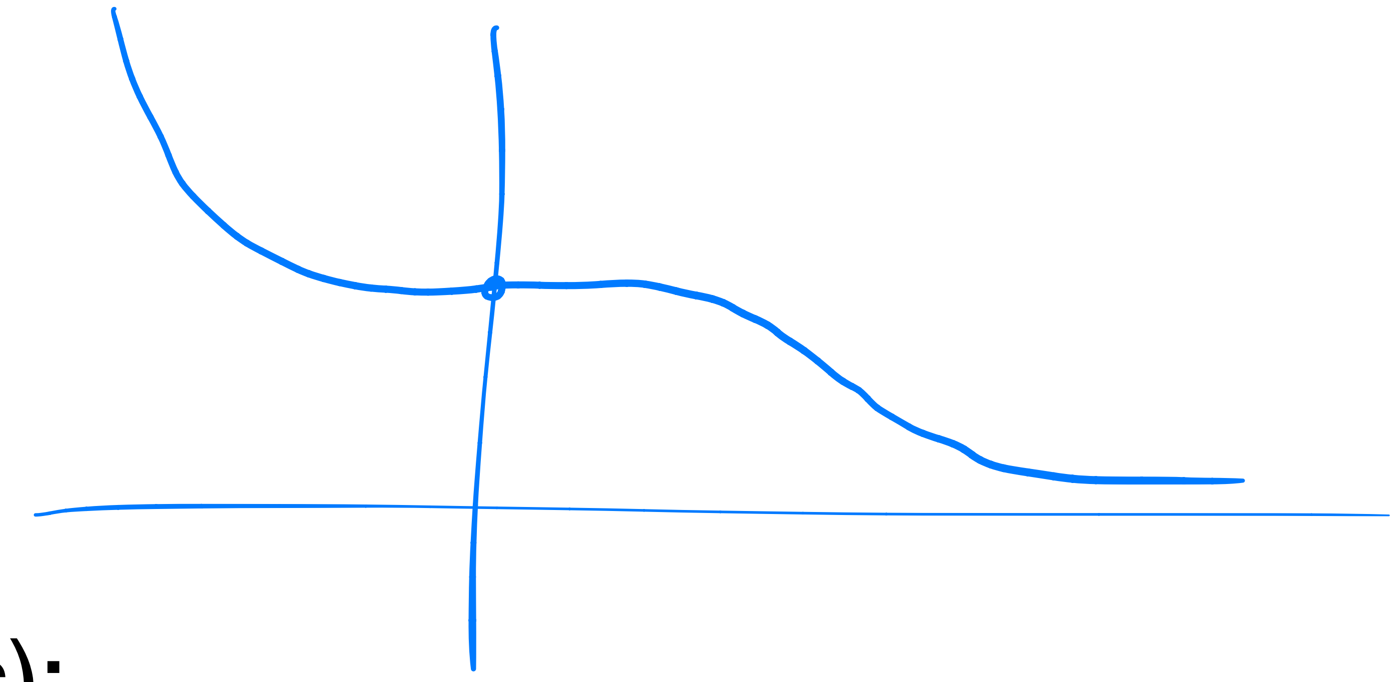

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- For non-convex functions, we could hope to find a local minima.

- What we can prove (under some regularity conditions) is a little weaker: Both GD (with some constant learning rate) and SGD (with some decaying learning rate) converge to a stationary point, i.e.

$$\text{As } k \to \infty, \ \nabla J(\theta^k) \to 0$$

# Today

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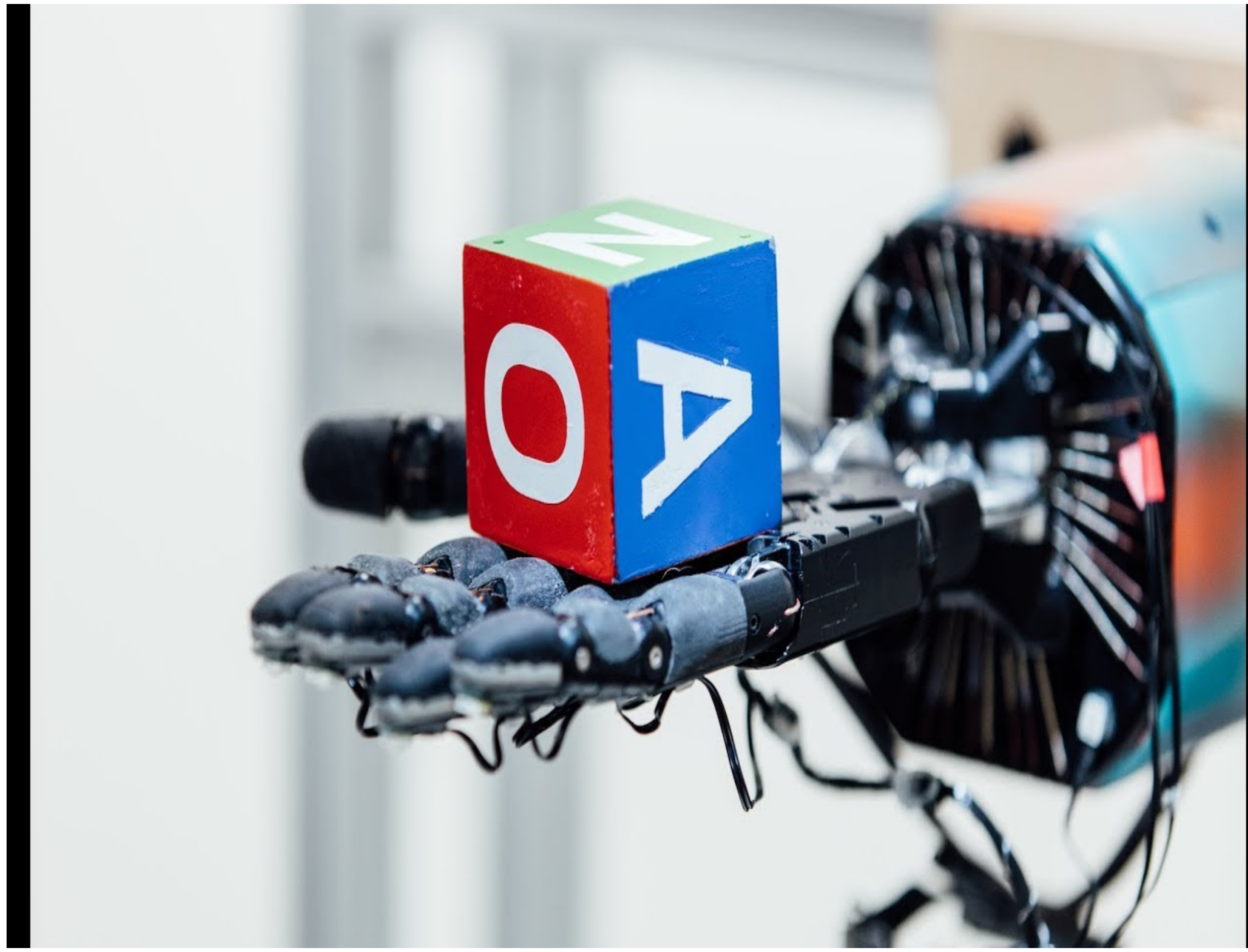
# Policy Optimization


[AlphaZero, Silver et.al, 17]


[OpenAI Five, 18]



[OpenAI,19]

# The Learning Setting:

We don't know the MDP, but we can obtain trajectories.

The Finite Horizon, Learning Setting. We can obtain trajectories as follows:

- We start at $s_0 \sim \mu$.
- We can act for $H$ steps and observe the trajectory $\tau = \{s_0, a_0, s_1, a_1, \ldots, s_{H-1}, a_{H-1}\}$

Note that with a simulator, we can sample trajectories as specified in the above.

# Optimization Objective

- Consider a parameterized class of policies:

$$\{\pi_\theta(a \mid s) \mid \theta \in \mathbb{R}^d\}$$

(why do we make it stochastic?)

- Objective $\max_\theta J(\theta)$, where

$$J(\theta) := \mathbb{E}_{s_0 \sim \mu}\left[V^{\pi_\theta}(s_0)\right] = \mathbb{E}_{\tau \sim \rho_{\pi_\theta}}\left[\sum_{h=0}^{H-1} r(s_h, a_h)\right]$$

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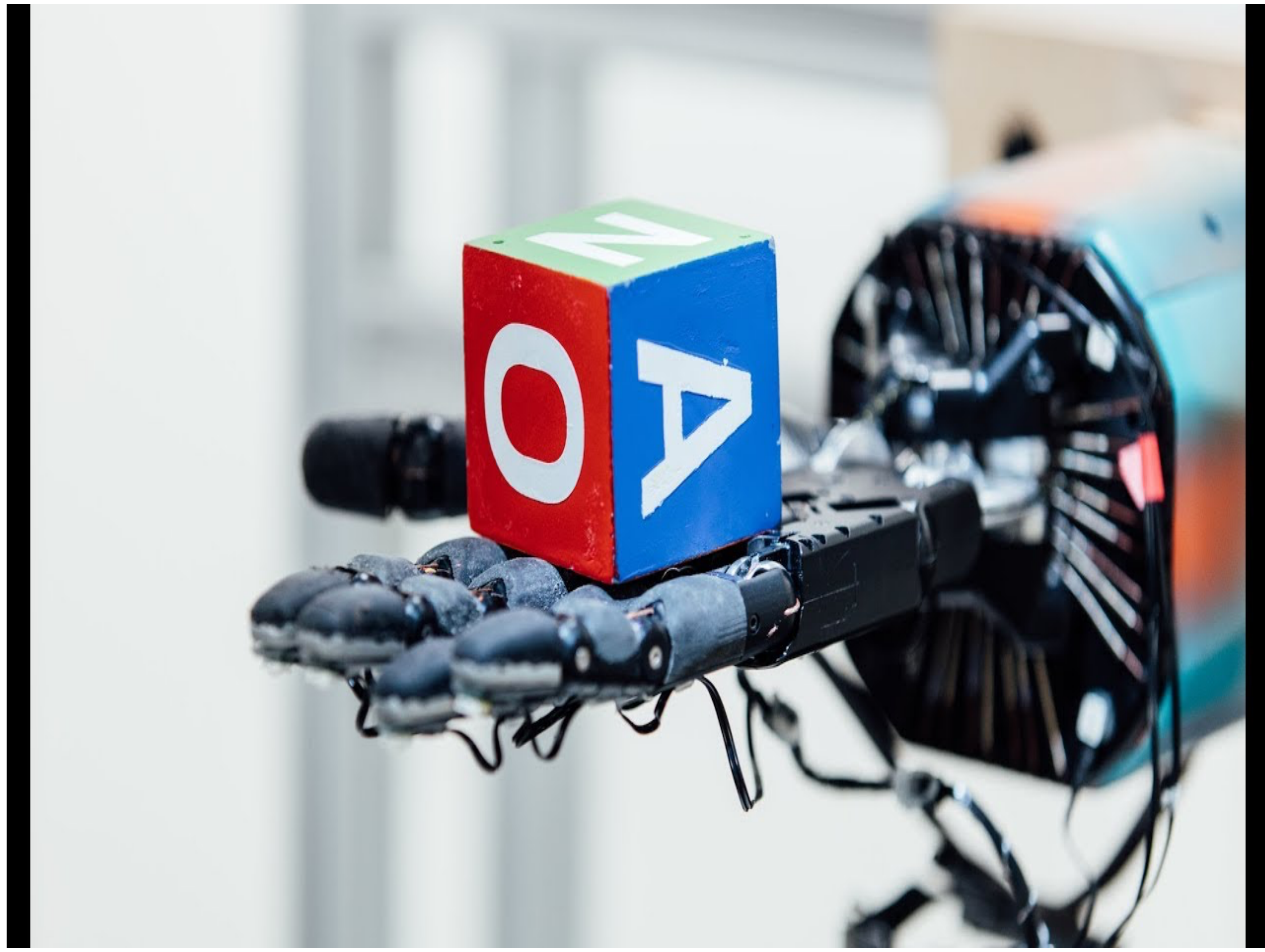
Main question for today's lecture:
how to compute the gradient?

# What are parameterized policies?


[AlphaZero, Silver et.al, 17]


[OpenAI Five, 18]



[OpenAI,19]

A state:

- Tabular case: an index in $[|S|] = \{1,\ldots|S|\}$
- Real world: a list/array of the relevant info about the world that makes the process Markovian.
  - e.g. sometimes make a feature vector $\phi(s, a, h) \in \mathbb{R}^d$ which we believe is a "good representation" of the world
  - we sometimes append history info into the current state

# Example Policy Parameterizations

Recall that we consider parameterized policy $\pi_\theta(\cdot \mid s) \in \Delta(A), \forall s$

## 1. Softmax linear Policy

Feature vector $\phi(s, a, h) \in \mathbb{R}^d$, and parameter $\theta \in \mathbb{R}^d$

$$\pi_\theta(a \mid s, h) = \frac{\exp(\theta^\top \phi(s, a, h))}{\sum_{a'} \exp(\theta^\top \phi(s, a', h))}$$

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**2. Neural Policy:**

Neural network $f_\theta : S \times A \times [H] \mapsto \mathbb{R}$

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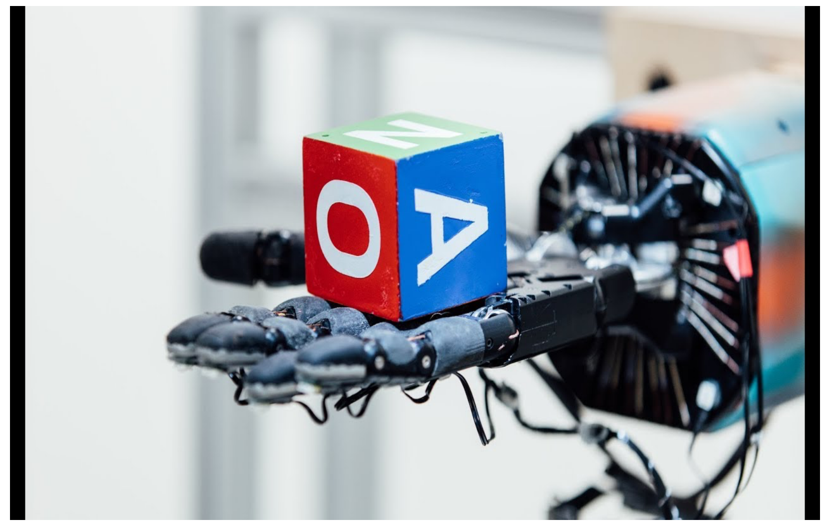
# Example Policy Parameterization for "Controls"



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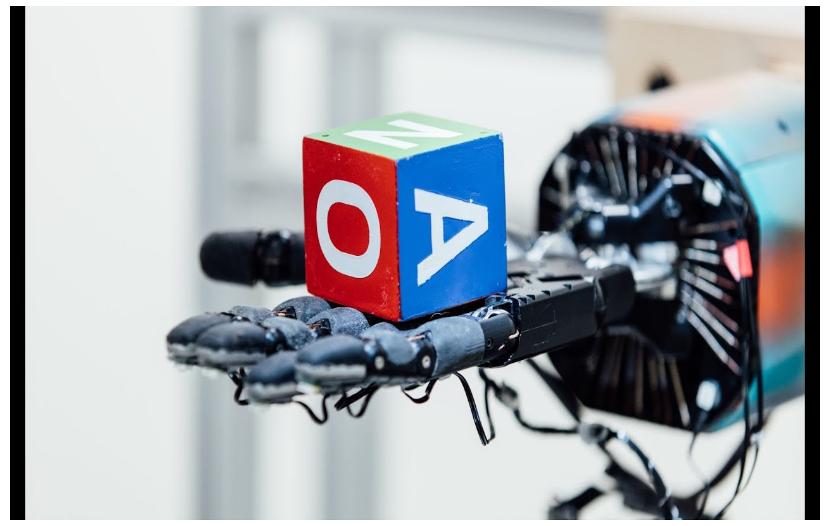


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- Feature vector: $\phi(s, h) \in \mathbb{R}^d$,
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  (and maybe $\sigma \in R^+$)

- Policy: sample action from a (multivariate) Normal with mean $\theta \cdot \phi(s, h)$ and variance $\sigma^2 I$, i.e.

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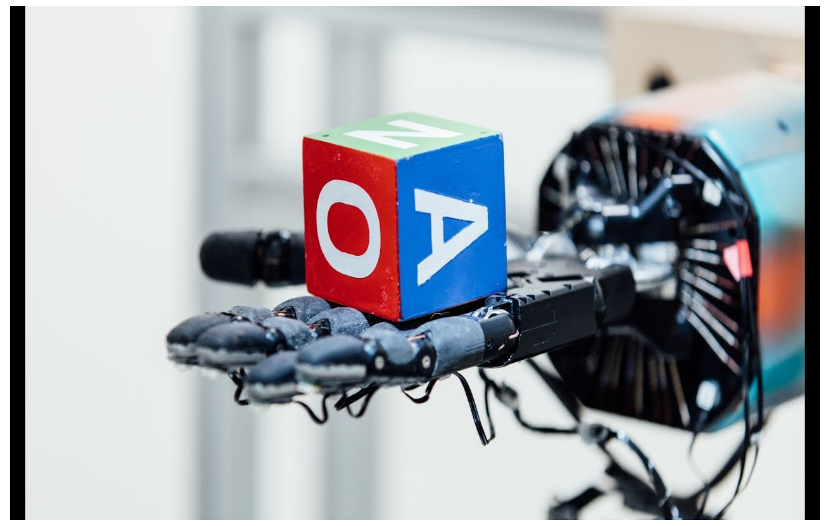

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  - In supervised learning, stochastic gradient was just gradient on one sample—will that work here?
  - Won't work: $\theta$-dependence is inside the distribution, not inside the expectation
  - So how can we unbiasedly estimate $\nabla_\theta J(\theta)$?
- Suppose we can compute $f(x)$, $P_\theta(x)$, and $\nabla P_\theta(x)$, and we can sample $x \sim P_\theta$
- We have that:

$$\nabla_\theta J(\theta) = \mathbb{E}_{x \sim P_\theta(x)}\left[\nabla_\theta \log P_\theta(x)\, f(x)\right]$$

Proof:

$$\nabla_\theta J(\theta) = \sum_x \nabla_\theta P_\theta(x) f(x)$$

$$= \sum_x P_\theta(x) \frac{\nabla_\theta P_\theta(x)}{P_\theta(x)} f(x)$$

# The Likelihood Ratio Method

- Suppose $J(\theta) = \mathbb{E}_{x \sim P_\theta}\left[f(x)\right] = \sum_x P_\theta(x) f(x)$, and our objective is $\max_\theta J(\theta)$.
- Computing $\nabla_\theta J(\theta)$ exactly may be difficult (due to the sum over $x$=trajectories)
  - So GD not an option—what about SGD?
  - In supervised learning, stochastic gradient was just gradient on one sample—will that work here?
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$\widehat{\nabla}_\theta J(\theta) = \nabla_\theta \log P_\theta(x) \cdot f(x)$, where $x \sim P_\theta$

# The Likelihood Ratio Method, continued

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- An unbiased estimate is given by:

$\widehat{\nabla}_\theta J(\theta) = \nabla_\theta \log P_\theta(x) \cdot f(x)$, where $x \sim P_\theta$

- We can lower variance by drawing $N$ i.i.d. samples from $P_\theta$ and averaging:

$$\widehat{\nabla}_\theta J(\theta) = \frac{1}{N} \sum_{i=1}^{N} \nabla_\theta \log P_\theta(x_i)\, f(x_i)$$

# Today

- ✓ Feedback from last lecture
- ✓ Recap+
- ✓ Gradient Descent (ok this is also sort of recap)
- ✓ Policy Gradient
- ✓ Likelihood ratio method

# Summary:

- Q-learning and TD(0) are online variants of fitted DP that use SGD
- PG approach: let's directly try to optimize the objective function of interest!

Attendance:
bit.ly/3RcTC9T

Feedback:
bit.ly/3RHtlxy